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Discovery

A representation of zarisky topology in a ring c(x)

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ABSTRACT

The aim of this paper is to construct a Zarisky topology in a ring C(X) through a functor C, where X is a pointed topological spaces and construct a representation of Zarisky topology in a ring C(X). In this paper we show that

i) Spec(C(X)) is a Zarisky topology in a ring C(X), where Spec(C(X)) denotes the set of all prime ideals of ring C(X) and construct a functor 'Spec' associated with Zarisky topology; and finally

ii) we study the functor 'Spec' associated with Zarisky topology;

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1. INTRODUCTION

The concept of affine algebraic sets plays an important role to construct Zarisky topology. In this section we give some basic definitions.

Definition 1.1

Let $K[x_1,x_2,x_3,\ldots,x_n]$ denotes the polynomial ring over an arbitrary field K in n-variables and $f_1, f_2, f_3, \ldots, f_m \in K[x_1,x_2,x_3,\ldots,x_n]$.

The function $V: S \to K^n$ defined by $V(S) = \{(x) \in K^n: f(x) = 0 , \forall f \in S\}$,
where $S = \{ f_1, f_2, f_3 ,......,f_m \} \subset K[x_1,x_2,x_3,....,x_n]$ and $(x) = (x_1,x_2,x_3,....,x_n) \in K^n$ is called the algebraic set i.e., the algebraic set V(S) is the set of solutions in K^n of the system of equations:
$f_1(x_1,x_2,x_3,....,x_n) = 0$, $f_2(x_1,x_2,x_3,....,x_n) = 0$, $f_3(x_1,x_2,x_3,....,x_n) = 0$,..., $f_m(x_1,x_2,x_3,....,x_n) = 0$.

Definition 1.2
A subset A in K^n is called an affine algebraic set if $A = V(S)$ for some $S \subset K[x]$.
Thus any algebraic set A is defined by a finite set of polynomials in K[x] .

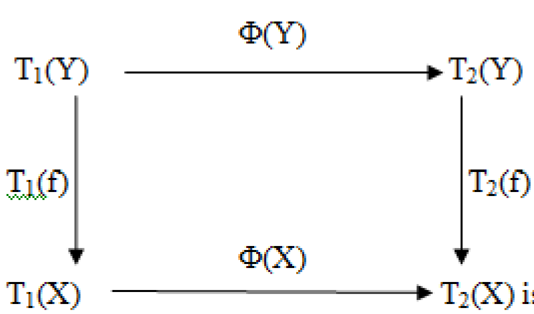
Definition 1.3
An algebraic set A in K^n is called irreducible or an affine variety iff $A \neq B \cup C$, where B and C are algebraic sets in K^n and $A \neq B$, $A \neq C$.

Definition 1.4
Let k be a subfield of K . If A is an affine algebraic set in K^n admits a set of generators in $k[x_1,x_2,x_3,....,x_n] \subset K[x_1,x_2,x_3,....,x_n]$,then A is called an affine (K,k) algebraic set and k is called the field of definition of A .
Thus an affine (K,k) algebraic set A is a subset in K^n consisting of all common zeros of a subset of polynomials in $k[x_1,x_2,x_3,....,x_n]$.
If $k = K$, we call A is an absolute affine algebraic set in K^n.

Definition 1.5
A category ***C*** consists of
a) a class of objects X,Y,...,denoted by Ob(***C***);
b) for each ordered pair of objects X,Y a set of morphisms with domain X and range Y denoted by ***C***(X,Y);
c) for each ordered triple of objects X,Y and Z and a pair of morphisms; $f: X\to Y$ and $g:Y\to Z$, their composite is denoted by $gf :X\to Z$, satisfying the two axioms:
i) associativity
ii) identity

Definition 1.6
Let ***C*** and ***D*** be two categories. A contravariant functor T from ***C*** to ***D*** consists of
a) an object function which assigns to every object X of ***C*** and object T(X) of ***D*** ; and
b) a morphism function which assigns to every morphism $f: X \to Y$ in ***C*** , a morphism
$T(f) : T(Y) \to T(X)$ in ***D*** such that
i) $T(I_X) = I_{T(X)}$,
ii) $T(g f) = T(f) T(g)$, for $g : Y\to Z$ in ***C***

Definition 1.7
Let **C** and **D** be categories. Suppose T_1 and T_2 are both contravariant functors from **C** to **D**.
A natural transformation Φ from T_1 to T_2 is a function from the objects of **C** to the morphisms of **D** such that for every morphism $f: X \to Y$ in **C** the following condition holds:
$\Phi(X) T_1(f) = T_2(f) \Phi(Y)$,i.e., the diagram



is commutative.

2. ELEMENTARY PROPERTIES

In this section we give some basic results which are essential in the sequel.

Lemma 2.1

A subset U of K^n is an open set iff K^n- U is an affine k – algebraic set.
Thus an affine k-algebraic sets in K^n are the closed sets in K^n.

Proof: Using [4] and [5] , it follows.

Lemma 2.2

Any subset A of K^n is an affine k-variety , then a subset A is closed in K^n iff it is an affine k-algebraic set.

Proof: Using **Definition 1.2**, **Definition 1.4** and **Lemma 2.1** , it follows.

Lemma 2.3

Any finite subset of K^n is an algebraic set.

Proof: Using [4, art.5.5] , it follows.

Lemma 2.4

An algebraic set A in K^n is affine variety iff I(A) is a prime ideal.

Proof: Using [4, art.5.5] , it follows.

Lemma 2.5

Let **C** be any category and **T** be a contravariant functor from **C** to **S** (category of sets and functions) .Then for any object C in **C** , there is an equivalence
Ω: ($\mathbf{h^C}$,**T**) $\rightarrow$**T**(C), where ($\mathbf{h^C}$,**T**) is the class of natural transformations from the set valued functor $\mathbf{h^C}$ to the set valued functor **T** such that Ω is natural in **C** and **T**.

Proof: From [4, appendix B] , it follows.

Lemma 2.6

For each subset P of a ring R , let X = V(P) denotes the set of all prime ideals of R which contains P, then

i) V(0) = X and V(1) = $\varnothing$,

ii) If $(P_i)_{i\in I}$ is any family of subsets of prime ideals of R, then

$$V\left(\bigcup_{i\in I} P_i\right) = \bigcap_{i\in I} V(P_i)$$

iii) V(P_1) $\cup$ V(P_2) = V($P_1 \cap P_2$) = V($P_1 P_2$).

Proof: Using [4, art.5.5] , it follows.
This shows that the set V(P) satisfies all the axioms for closed sets in a topological space.

3. FUNCTOR ASSOCIATED WITH ZARISKY TOPOLOGY

In this section we construct and investigate the **Zarisky topology** . To do this we prove the following:

Theorem 3.1

All affine k-algebraic sets in K^n are the closed sets in K^n . Then this sets form a topology in K^n. This topology is called the **Zarisky topology** in K^n.

Proof: Since empty set and whole set are closed sets ; the intersection of any family of closed sets is a closed set; and the union of two closed set is a closed set. Using **Lemma 2.2,** it follows.

Let X be a pointed topological space and C(X) be the set of all base point preserving real valued continuous functions defined on X. Then (C(X), +,·) forms a ring, where (f+g)(x) = f(x) + g(x) and (f·g)(x) = f(x)·g(x).

Let X be a pointed topological spaces and construct a representation of **Zarisky topology** in a ring **C(X).**

Let **Top** denote the category of pointed topological spaces and base point preserving continuous maps and **R** be the category of rings and ring homomorphisms , then by [7]

Theorem 3.2 C: **Top** → **R** is a contravariant functor, where **Top** denote the category of pointed topological spaces and base point preserving continuous maps and **R** denote the category of rings and ring homomorphisms.

Let C(X) be a ring. Define prime spectrum **Spec**(C(X)) of C(X) by
Spec(C(X)) = {P : P is prime ideal of C(X)}.

Theorem 3.3
Let **Spec**(C(X)) denote the set of all prime ideals of a ring C(X). All subsets in the set of all prime ideals of a ring C(X) , form a topology **Spec**(C(X)) , is called **Zarisky topology**.

Proof: Using **Lemma 2.2**,**Lemma 2.6 & Theorem 3.1** , it follows.

Theorem 3.4
Let **R** be the category of rings and ring homomorphisms and **T** denote the category of sets and functions. Then **Spec** : **R**→**T** is a contravariant functor

Proof: Let C(X) and C(Y) are rings in **R** and f: C(X)→C(Y) is a ring homomorphism in **R** . Define f^*: Spec(C(Y))→Spec(C(X)) by $f^*(Q) = f^{-1}(Q)$, ∀ Q∈Spec(C(Y)) , then $f^{-1}(Q)$ is a prime ideal of C(X) and hence $f^{-1}(Q)$ ∈ Spec(C(X)).
Let f : C(X) → C(Y) and g : C(Y)→C(Z) be a ring homomorphisms in **R** .Then gf : C(X)→C(Z) is also a ring homomorphism.
Therefore **Spec**(gf): **Spec**(C(Z)) → **Spec**(C(X)) by
Spec(gf)(Q) = $(gf)^{-1}(Q)$,∀ Q∈ **Spec**(C(Z))
= $(f^{-1}g^{-1})(Q)$
= **Spec**(f) ((**Spec**(g)(Q))
⇒ **Spec**(gf)(Q) = (**Spec**(f) **Spec**(g))(Q) , ∀ Q∈ **Spec**(C(Z)).
⇒ **Spec**(gf) = **Spec**(f) **Spec**(g)

Also, **Spec**($I_{C(X)}$) = $I_{\mathbf{Spec}(C(X))}$, where $I_{C(X)}$: C(X)→ C(X) in **R**.

Using Lemma 2.5 , we say that the set of pointed topological space X there is a bijective correspondence with the set of all prime ideals of the ring C(X), where C(X) be the set of all base point preserving real valued continuous functions defined on X.

Let $h^{\mathbf{C(X)}}$(**C(**Y**)**) = Hom(**C(**Y**)**,**C**(X)).

We define for each f: **C**(X) → **C**(Y) in **R**, $h^{\mathbf{R}[A]}$(f) = Hom(**C**(Y), **C(**X**)**) → Hom(**C**(X),**C**(X)) by
$h^{\mathbf{C(X)}}(f)(\alpha) = \alpha \text{ o } f$, ∀ α ∈ Hom(**C**(Y),**C**(X).

Theorem 3.5 $h^{\mathbf{C}(X)}$: **R**→**T** is a contravariant functor.
Proof: From [5] , it follows

Thus we have two contravariant functors **Spec** and $\mathbf{h}^{\mathbf{C(X)}}$ from the category **R** to the category **T**.

Now we have the following **Theorems**:

Theorem 3.6

For each set X in **Top** , there is an equivalence

Ω : ($\mathbf{h}^{\mathbf{C(X)}}$,**Spec**)→ **Spec**(C(X)), where ($\mathbf{h}^{\mathbf{C(X)}}$,**Spec**) is the set of all natural transformations from the contravariant functor '**Spec'** to the contravariant functor $\mathbf{h}^{\mathbf{C(X)}}$**.**

Proof: Using Lemma 2.5, it follows.

Corollary 3.7

For each set pointed topological space X **,** there is an equivalence from the set of all natural transformations from the contravariant functor **Spec** to the contravariant functor $\mathbf{h}^{\mathbf{C(X)}}$ to the **Zarisky topology** '**Spec(**C(X)**)'**

Proof: Using **Lemma 2.5**, **Lemma 2.8**, **Theorem 3.1, Theorem 3.3** ,**Theorem 3.4** and **Theorem 3.5**, it follows.

Proposition 3.8

The set ($\mathbf{h}^{\mathbf{C(X)}}$,**Spec**) of all natural transformations from the contravariant functor **Spec** to the contravariant functor $\mathbf{h}^{\mathbf{C(X)}}$ froms a **Zarisky topology** in **C**(X).

Proof: Using **Lemma2.5**,**Theorem 3.4, Theorem 3.5** and **Corollary 3.7**, it follows.

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