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A study of generalised hypergeometric functions with certain $\overline{H}$-functions of one and two variables

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A STUDY OF GENERALISED HYPERGEOMETRIC FUNCTIONS WITH CERTAIN $\overline{H}$-FUNCTIONS OF ONE AND TWO VARIABLES

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CHAPTER I

INTRODUCTION

The following functions are needed mainly in our work:

The hypergeometric function is defined and represented in the following form:

$$ {}_2F_1(a,b;c;z) = \sum_0^{\infty} \frac{(a)_n (b)_n}{(1)_n (c)_n} z^n \qquad (1.1)$$

Where c is neither zero nor a negative integer, where in (1.1), the pochhammer's symbol employed is defined as

$$(\alpha)_n = \alpha(\alpha+1)(\alpha+2).......(\alpha+n-1), n \geq 1,\ (\alpha)_0 = 1$$

In the year 1908, Branes defined the hyper geometric function in terms of a Mellin-type integral deviating from the conventional method of defining a special function in terms of an infinite series, in the integral form:

$$ {}_2F_1(a,b,c;-z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{\Gamma(-s)\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)} z^s ds \qquad (1.2)$$

Where $i = \sqrt{-1}$.

Where the pole of $\Gamma(-s)$, at the point s=0,1,2,3,……………, are separated from those of $\Gamma(a+s)$, at the points $s = -a - v_1 (v_1 = 0,1,2,......)$ and $\Gamma(b+s)$ at the points $s = -b - v_2 (v_2 = 0,1,2,......)$ And $|\arg(z)| < \pi$

One of the importance of this definition lies in the fact that the Millen transform of ${}_2F_1(.)$ is the coefficient of z^{-s} in the integrand of (1.2.), that is

$$\int_0^{\infty} z^{s-1} {}_2F_1(a,b,c;-z) dz = \frac{\Gamma(c)\Gamma(s)\Gamma(a-s)\Gamma(b-s)}{\Gamma(a)\Gamma(b)\Gamma(c-s)}, \qquad (1.3)$$

Where $\mathrm{Re}(s) > 0, \mathrm{Re}(a-s) > 0, \mathrm{Re}(b-s) > 0.$

Swaroop (1964) introduced and studied the hypergeometric function transform; that kernel is the Gauss's hyper geometric function. Saxena,(1967b) and Kalla and Saxena (1969) used the hypergeometric function in defining certain fractional integration operators. Mathai and saxena (1966) introduced the probability function associated with a ${}_2F_1$(a,b;c; .).

Generalized hypergeometric function is defined in the form:

$$ {}_pF_q\left[\begin{matrix}\alpha_1,...,\alpha_p; \\ \beta_1,...,\beta_q;\end{matrix} z\right] = F\left[\begin{matrix}(\alpha_p) \\ (\beta_q)\end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (\alpha_j)_n}{\prod_{j=1}^{q} (\beta_j)_n} \frac{z^n}{n!}; \qquad (1.4)$$

In which no denominator parameter is allowed to take zero or a negative integer value. If any of the parameter α_p is zero or a negative integer, the series terminates. The function defined by (1.4) will be denoted briefly by ${}_pF_q(z)$.

The Mellin-Branes integral for ${}_pF_q(z)$ is

$$
{}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p; \\ \beta_1,\ldots,\beta_q;\end{matrix} z\right] =
$$

$$
\frac{\prod_{j=1}^{p}\Gamma(\beta_j)}{\prod_{j=1}^{q}\Gamma(\alpha_j)} \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{\prod_{j=1}^{p}\Gamma(\alpha_j+s)\Gamma(-s)}{\prod_{j=1}^{q}\Gamma(\beta_j+s)}(-z)^s\, ds \tag{1.5}
$$

Where $\mathrm{i} = \sqrt{-1}$ and for convergence, $\mathrm{p} \le \mathrm{q}$ or $\left(\mathrm{p} = \mathrm{q}+1 \text{ and } |\mathrm{z}| < 1\right)$, $\left|\arg\left(-\mathrm{z}\right)\right| < \pi$

When $\mathrm{p} > \mathrm{q}+1$ the series in (1.4) diverges. The path of integration is indented, if necessary, in such a manner that the poles of $\Gamma(-s)$ at the points $s = 0,1,2,\ldots;$ are separated from those of $\Gamma(\alpha_j + s)$, at the points $\alpha_j = -s - v_j$; $v_j = 0,1,2,\ldots;\mathrm{j} = 1,\ldots,p$. An empty product is interpreted as unity.

It is a matter of common knowledge that the Gaussian hypergeometric function ${}_2F_1(\mathrm{a},\ \mathrm{b};\ \mathrm{c};\ 1)$ can be summed up as $\dfrac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$, when $\mathrm{Re}\left(\mathrm{c}-\mathrm{a}-\mathrm{b}\right) > 0$ and that this formula is one of the simplest case of an assumable hypergeometric series which has found much use in the simplification of many problems.

In order to give a meaning to the symbol ${}_pF_q(.)$ When $\mathrm{p} > \mathrm{q}+1$, MacRobert (1937-1941) defined and studied his E-function in the form

$$
E(p;a_r;q;\rho_B:x) = \frac{\Gamma(a_q+1)}{\Gamma(\rho_1-a_1)\Gamma(\rho_2-a_2)\ldots\Gamma(\rho_q-a_q)}
$$

$$
.\prod_{u=1}^{q}\int_0^\infty \lambda_u^{\rho_u-a_u-1}(1+\lambda_u)^{-\rho_u}\, d\lambda_u
$$

$$
.\prod_{v=2}^{p-q-1}\int_0^\infty e^{-\lambda_{q+v}}\lambda_{q+v}^{a_{q+v}-1}d\lambda_{q+v}\int_0^\infty e^{-\lambda_p}
$$

$$
.\lambda_p^{a_p-1}\left[1+\frac{\lambda_{q+2}\lambda_{q+3}\ldots\lambda_p}{(1+\lambda_1)(1+\lambda_2)\ldots(1+\lambda_q)x}\right]^{-a_{q+1}} d\lambda_p \tag{1.6}
$$

A detailed account of this function can be found in Erdelyi et al. (1953).

Meijer (1946) introduced a generalization of the E - function in the form:

$$
G_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix}(a_p)\\(b_q)\end{matrix}\right] = G_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix}a_1,\ldots,a_p\\b_1,\ldots,b_q\end{matrix}\right]
$$

$$= \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j - s) \prod_{j=1}^{n} \Gamma(1 - a_j + s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + s) \prod_{j=n+1}^{p} \Gamma(a_j - s)} z^s ds \qquad (1.7)$$

Where L is a suitable contour separating the poles of $\Gamma(b_j - s)$ for j $=1, \ldots, m$

From those of $\Gamma(1 - a_j + s)$ for j $=1, \ldots, n$. The poles of theintegrand are assumed to be simple.

In an attempt to discover, the solution of certain integral equations Saxena, V. P. (1982) introduced the I - function in the following form:

The I-function introduced by Saxena [6] will be represented and defined as follows :

$$I[Z] = I^{m,n}_{p_i,q_i;r}[Z] = I^{m,n}_{p_i,q_i:r}\left[z \middle| \begin{matrix} (a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i} \\ (b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i} \end{matrix} \right] = \frac{1}{2\pi\omega} \int_L \chi(\xi) d\xi \qquad (1.8)$$

where $\omega = \sqrt{-1}$

$$\chi(\xi) = \frac{\prod_{j=1}^{m} \Gamma(b_j - \beta_j \xi) \prod_{j=1}^{n} \Gamma(1 - a_j + \alpha_j \xi)}{\sum_{i=1}^{r} \left\{ \prod_{j=m+1}^{q_i} \Gamma(1 - b_{ji} - \beta_{ji}) \prod_{j=n+1}^{p_i} \Gamma(a_{ji}, \alpha_{ji}) \right\}} \qquad (1.9)$$

$p_{i,} q_i (i = 1, \ldots, r), m, n$ are integers satisfying $0 \le n \le p_i$, $0 \le m \le q_i, (i = 1, \ldots, r), r$ is finite $\alpha_j, \beta_j, \alpha_{ij}, \beta_{ji}$ are real and $a_{j,} b_j, a_{ji}, b_{ji}$ are complex numbers such that

$\alpha_j (b_h + v) \ne \beta_h (a_j - v - k)$ for $v, k = 01, 2, \ldots$

L is a contour which runs from $\sigma - w^{\infty}$ to $\sigma + w^{\infty}$ (σ is real),

s =(a$_j$-1-υ)/α_j; j= 1, 2, . . . ,n; υ= 0, 1, 2, ...

$s = (b_j + \upsilon) / \beta_j$; j= 1, 2, . . . , m ; υ= 0, 1, 2, ...

Lie to the L.H.S. and R.H.S. of L , respectively.

I -function reduces to H -function, when r = 1.

The relation between I -and H -function is given below:

$$I^{m,n}_{p_i,q_i;1}\left[z \,\middle|\, \begin{matrix} \{(a_j,\alpha_j)_{1,n}\},\{(a_{ji},\alpha_{ji})_{n+1,p_i}\} \\ \{(b_j,\beta_j)_{1,m}\},\{(b_{ji},\beta_{ji})_{m+1,q_i}\} \end{matrix} \right]$$

$$= H^{m,n}_{p_1,q_1}\left[z \,\middle|\, \begin{matrix} (a_1,\alpha_1),(a_2,\alpha_2),\ldots,(a_{p_1},\alpha_{p_1}) \\ (b_1,\beta_1),(b_2,\beta_2),\ldots,(b_{q_1},\beta_{q_1}) \end{matrix} \right] \qquad (1.10)$$

Viashya, Jain and Verma (1989) found certain identity, multiplication theorems differentiation formulae and some integrals involving the I -function (1.8).

Agarwal (1965) extended the Meijer's G-function to G-function of two variables. The work of agarwal, (1965) and Sharma, (1965) gave a fresh impetus to numerous workers to further generalize the G-symbol to G- function of n- variables due to Khadia(1970). The G-function of n-variables was further converted to the H-symbol of n- variables by Saxena (1974,77) further generalized the H-function of n- variables into the multivariable I- function .

In all these, aforesaid G-, H- and I- type function of one two and -n variables, the coefficients of the variable of integration in the gamma function products of the integrand (of the integrals defining the functions) were taken to be real positive. Considering these multipliers as complex number quite a few papers have appeared in the literature.

The aforesaid generalised hypergeometric function have also been studied by Gupta and Rathie (1968), Khadia and Goyal (1975), Love (1967), Srivastava and Buschman (1972), Bora and Saxena (1971), Barnes (1908), Bajpai and AL-Hawaj (1989) through various important results with generalized hypergeometric functions.

Pandey and Pandey (1985) have obtained power series expansion for the modified H- function of several variables, which by assigning suitable value of the parameters give rise to the power series expansions given by Lawricella and other.

MULTIVARIABLE H-FUNCTION

When $n_2 = n_3 = \dots = n_{r-1} = 0 = p_2 = p_3 = \dots = p_{r-1}$ and $q_2 = q_3 = \dots = q_{r-1} = 0$; multivariable I- function reduces to the H- function of several variables due to Srivastava and Panda (1976) defined in the following manner, which it self is a generalization of the H- function of several variables due to Saxena, (1974).

$$H[z_1,\dots,z_r] = H^{0,n:m_1,n_1;\dots;m_r,n_r}_{p,q:p_1,q_1;\dots;p_r,q_r}$$

$$\cdot\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\left|\begin{matrix}(a_j;A'_j,\dots,A_j^{(r)})_{1,p}:(c'_j,C'_j)_{1,p_1};\dots;(c_j^{(r)},C_j^{(r)})_{1,p_r} \\ (b_j;B'_j,\dots,B_j^{(r)})_{1,q}:(d'_j,D'_j)_{1,q_1};\dots;(d_j^{(r)},D_j^{(r)})_{1,q_r}\end{matrix}\right.\right]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\dots\int_{L_r}\Phi(s_1,\dots,s_r)\theta_1(s_1)\dots\theta_r(s_r)z_1^{s_1}\dots z_r^{s_r}.ds_1\dots ds_r \qquad (1.11)$$

Where $\omega = \sqrt{-1}$;

$$\Phi(s_1,\dots,s_r) = \frac{\prod_{j=1}^{n}\Gamma(1-a_j+\sum_{i=1}^{r}A_j^{(i)}s_i)}{\prod_{j=n+1}^{p}\Gamma(a_j-\sum_{i=1}^{r}A_j^{(i)}s_i)\prod_{j=1}^{q}\Gamma(1-b_j+\sum_{i=1}^{r}B_j^{(i)}s_i)} \qquad (1.12)$$

$$\theta_i(s_i) = \frac{\prod_{j=1}^{m_i}\Gamma(d_j^{(i)}-D_j^{(i)}s_i)\prod_{j=1}^{n_i}\Gamma(1-c_j^{(i)}+C_j^{(i)}s_i)}{\prod_{j=m_i+1}^{q_i}\Gamma(1-d_j^{(i)}-D_j^{(i)}s_i)\prod_{j=n_i+1}^{p_i}\Gamma(c_j^{(i)}-C_j^{(i)}s_i)} \quad ,$$

$$\forall i \in \{1,\dots,r\} \qquad (1.13)$$

In (1.11) the superscript (i) stands for the number of prime, e.g., $b^{(1)} = b'$, $b^{(2)} = b''$ and so on; and an empty product is interpreted as unity. Further it is assumed that the parameters

$$\begin{Bmatrix} a_j, \text{j}= 1,\ldots,\text{p}; c_j^{(i)}, \text{j}= 1,\ldots, p_i; \\ b_j, \text{j}= 1,\ldots,q; d_j^{(i)}, \text{j}= 1,\ldots,q_i; \end{Bmatrix}, \quad \forall \text{i} \in \{1, \ldots, \text{r}\}$$

Are complex numbers, and the associated coefficients

$$\begin{Bmatrix} A_j^{(i)}, \text{j}= 1,\ldots,\text{p}; C_j^{(i)}, \text{j}= 1,\ldots, p_i; \\ B_j^{(i)}, \text{j}= 1,\ldots,q; D_j^{(i)}, \text{j}= 1,\ldots,q_i; \end{Bmatrix}, \quad \forall \text{i} \in \{1, \ldots, \text{r}\}$$

Are positive real number such that

$$\Omega_i = \sum_{j=1}^{p} A_j^{(i)} + \sum_{j=1}^{p_i} C_j^{(i)} - \sum_{j=1}^{q} B_j^{(i)} - \sum_{j=1}^{q_i} D_j^{(i)} \leq 0, \tag{1.14}$$

and

$$\Lambda_i = -\sum_{j=n+1}^{p} A_j^{(i)} + \sum_{j=1}^{n_i} C_j^{(i)} - \sum_{j=n+1}^{p_i} C_j^{(i)} - \sum_{j=1}^{q} B_j^{(i)}$$
$$+\sum_{j=1}^{m_i} D_j^{(i)} - \sum_{j=m_i+1}^{q_i} D_j^{(i)} > 0, \quad \forall \text{i} \in \{1, \ldots, \text{r}\} \tag{1.15}$$

Where the integrals n, p, m_i, n_i, p_i and q_i are constrained by the inequalities $0 \leq n \leq p, q \geq 0, 1 \leq m_i \leq q_i$ and $0 \leq n_i \leq p_i$, $\forall \text{i} \in \{1, \ldots, \text{r}\}$ and the equalities in (1.2.4) hold for suitably restricted values of the complex variables $z_1, \ldots, z_r$. The poles of the integrand in (1.11) are assumed to be simple. The contour L_i in the complex s_i-plane is of the Mellin-Branes type which runs from $\omega^{-\infty}$ to $\omega^{+\infty}$ with indentations, if necessary, to ensure that all the poles of $\Gamma(d_j^{(i)} - D_j^{(i)} s_i)$, j = 1, ..., m_i, are separated from those of $\Gamma(1-\tau_j^{(i)} + C_j^{(i)} s_i)$, j = 1, ..., n_i, and $\Gamma(1-a_j + \sum_{i=1}^{r} A_j^{(i)} s_i)$, j = 1, ..., n; $\forall i \in \{1, \ldots, \text{r}\}$

The multivariable H - function (1.11) converges absolutely under the conditions (1.2.5) for

$$|\arg z_i| < \frac{1}{2}\Lambda_i \pi, \quad \forall i \in \{1, \ldots, \text{r}\}, \tag{1.16}$$

The asymptotic expansion of algebraic order for the multivariable H-function, which will need in the analysis, is given below:

$$H[z_1, \ldots, z_r] = \begin{Bmatrix} 0(|z_1|^{A_1} \ldots |z_r|^{A_r}), \max\{|z_1|, \ldots, |z_r|\} \to 0 \\ 0(|z_1|^{B_1} \ldots |z_r|^{B_r}), n = 0, \min\{|z_1|, \ldots, |z_r|\} \to 0 \end{Bmatrix} \tag{1.17}$$

For $\text{i} = 1, \ldots, \text{r}$, with

$$\begin{Bmatrix} A_i = \min \operatorname{Re}(d_j^{(i)} / D_j^{(i)}), j = 1, \ldots\ldots\ldots, m_i \\ B_i = \max \operatorname{Re}[(c_j^{(i)} - 1) / C_j^{(i)}], j = 1, \ldots, n_i \end{Bmatrix} \tag{1.18}$$

Provided that each of the inequalities in (1.14), (1.15) and (1.16) hold.

If $A_j^{'} = \ldots = A_j^{(r)}$, $j = 1, \ldots, p$; $B_j^{'} = \ldots = B_j^{(r)}$, $j = 1, \ldots, q$ in (1.11),we get a special multivariable H-function studied by Saxena, (1974). On the other hand, if all of the capital letters are chosen to be one, the H-function of several variables defined by (1.11) reduces to the corresponding G-function of several variables studied by Khadia and Goyal (1970).

Generalized H-function as a symmetrical Fourier Kernel has been studied by Saxena and Modi (1975).

MULTIVARIABLE I-FUNCTION

The multivariable I - function introduced by Prasad (1986) will be defined and represented in the following manner:

$$I[z_1,...,z_r] = I^{0,q_2:...;0,n_r:(m',n');...;(m^{(r)},n^{(r)})}_{p_2,q_2:...;p_r,q_r:(p',q');...;(p^{(r)},q^{(r)})}$$

$$.\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \middle| \begin{matrix} (a_{2j};\alpha'_{2j};\alpha''_{2j})_{1,p_2} :...:(a_{rj};\alpha'_{rj},...,\alpha^{r}_{rj})_{1,p_r} :(a'_j,\alpha'_j)_{1,p'};...;(a^{(r)}_j,\alpha^{(r)}_j)_{1,p^{(r)}} \\ (b_{2j};\beta'_{2j};\beta''_{2j})_{1,q_2} :...:(b_{rj};\beta'_{rj},...,\beta^{r}_{rj})_{1,q_r} :(b'_j,\beta'_j)_{1,q'};...;(b^{(r)}_j,\beta^{(r)}_j)_{1,q^{(r)}} \end{matrix}\right]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}....\int_{L_r}\phi_1(s_1)...\phi_r(s_r)\psi(s_1,...,s_r)\,.z_1^{s_1}...z_r^{s_r}\,ds_1...ds_r \qquad (1.19)$$

Where $\omega = \sqrt{-1}$;

$$\phi_i(s_i) = \frac{\prod_{j=1}^{m^{(i)}}\Gamma(b_j^{(i)} - \beta_j^{(i)}s_{(i)})\prod_{j=1}^{n^{(i)}}\Gamma(1 - a_j^{(i)} + \alpha_j^{(i)}s_{(i)})}{\prod_{j=m^{(i)}+1}^{q^{(i)}}\Gamma(1 - b_j^{(i)} + \beta_j^{(i)}s_{(i)})\prod_{j=n^{(i)}+1}^{p^{(i)}}\Gamma(a_j^{(i)} - \alpha_j^{(i)}s_{(i)})};$$

$$\forall i \in \{1,\ ....,\mathrm{r}\} \qquad (1.20)$$

$$\psi(s_1,....s_r) = \frac{\prod_{j=1}^{n_2}\Gamma(1-a_{2j}+\sum_{i=1}^{2}\alpha_{2j}^{(i)}s_i)\prod_{j=1}^{n_3}\Gamma(1-a_{3j}+\sum_{j=1}^{3}\alpha_{3j}^{(i)}s_i)}{\prod_{j=n_2+1}^{p_2}\Gamma(a_{2j}-\sum_{i=1}^{2}\alpha_{2j}^{(i)}s_i)\prod_{j=n_3+1}^{p_3}\Gamma(a_{3j}-\sum_{i=1}^{3}\alpha_{3j}^{(i)}s_i)}$$

$$.\frac{....\prod_{j=1}^{n_r}\Gamma(1-a_{rj}+\sum_{i=1}^{r}\alpha_{rj}^{(i)}s_i)}{...\prod_{j=n_r+1}^{p_r}\Gamma(a_{rj}-\sum_{i=1}^{r}\alpha_{rj}^{(i)}s_i)\prod_{j=1}^{q_2}\Gamma(1-b_{2j}+\sum_{i=1}^{2}\beta_{2j}^{(i)}s_i)}$$

$$\frac{1}{....\prod_{j=1}^{q_r}\Gamma(1-b_{rj}+\sum_{i=1}^{r}\beta_{rj}^{(i)}s_i)} \qquad (1.21)$$

$\alpha_j^{(i)},\beta_j^{(i)},\alpha_{kj}^{(i)},\beta_{kj}^{(i)}$ $(\mathrm{i}=1,\ ....,\mathrm{r})(\mathrm{k}=2,\ ....,\mathrm{r})$ are positive numbers, $a_j^{(i)},b_j^{(i)},(\mathrm{i}=1,\ ....,\mathrm{r})$ a_{kj},b_{kj} $(\mathrm{k}=2,\ ....,\mathrm{r})$ are complex numbers and here $m^{(i)},n^{(i)},p^{(i)},q^{(i)}$ $(\mathrm{i}=1,\ ....,\mathrm{r})$ $,n_k,p_k,q_k$ $(\mathrm{k}=2,\ ....,\mathrm{r})$ are non negative integers where $0 \le m^{(i)} \le q^{(i)}$ $, 0 \le n^{(i)} \le p^{(i)}$, $q_k \ge 0$, $0 \le n_k \le p_k$. Here (i) denotes the numbers of the contours.

L_i In the complex s_i-plane of the Mellin Barnes type which runs from $-w^{\infty} to + w^{\infty}$

With indentation, if necessary to ensure that all the poles of $\Gamma(b_j^{(i)}-\beta_j^{(i)}s_i)$ (j = 1, ... , $m^{(i)}$) are separated from those of $\Gamma(1-a_j^{(i)}+\alpha_j^{(i)}s_i)$ (j = 1, ... , $n^{(i)}$), $\Gamma(1-a_{2j}+\sum_{i=1}^{2}\alpha_{2j}^{(i)}s_i)$ (j = 1, ... , n_2), ... , $\Gamma(1-a_{rj}+\sum_{i=1}^{r}\alpha_{rj}^{(i)}s_i)$ (j = 1, ... , n_r).

According to the asymptotic expansion of the gamma function, the counter integral (1.19) is absolutely convergent provided that

$$|\arg z_i| < \frac{1}{2}\pi U_i, U_i > 0 \; ; \quad i = 1, 2, \ldots, r \tag{1.22}$$

Where

$$U_i = \sum_{j=1}^{n^i}\alpha_j^{(i)} - \sum_{j=n^{(i)}+1}^{p^{(i)}}\alpha_j^{(i)} + \sum_{j=1}^{m^i}\beta_j^{(i)} - \sum_{j=m^{(i)}+1}^{q^{(i)}}\beta_j^{(i)}$$
$$+(\sum_{j=1}^{n_2}\alpha_{2j}^{(i)} - \sum_{j=n_2+1}^{p_2}\alpha_{2j}^{(i)}) + (\sum_{j=1}^{n_3}\alpha_{3j}^{(i)} - \sum_{j=n_3+1}^{p_3}\alpha_{3j}^{(i)})$$
$$+\ldots+(\sum_{j=1}^{n_r}\alpha_{rj}^{(i)} - \sum_{j=n_r+1}^{p_r}\alpha_{rj}^{(i)})$$
$$-(\sum_{j=1}^{q_2}\beta_{2j}^{(i)} + \sum_{j=1}^{q_3}\beta_{3j}^{(i)} + \ldots + \sum_{j=1}^{q_r}\beta_{rj}^{(i)}) \tag{1.23}$$

The asymptotic expansion of the I-function has been discussed by Prasad (1986). His results run as follow:

$I[z_1,\ldots,z_r] = 0(|z_1|^{\alpha_1}\ldots|z_r|^{\alpha_r}), \max\{|z_1|,\ldots,|z_r|\} \to 0$

Where

$$\alpha_i = \min \mathrm{Re}(b_j^{(i)}/\beta_j^{(i)}),\ j = 1, \ldots, m^{(i)} \; ; \; i = 1, \ldots, r \tag{1.24}$$

And $I[z_1,\ldots,z_r] = 0(|z_1|^{\beta_1}\ldots|z_r|^{\beta_r}), \min\{|z_1|,\ldots,|z_r|\} \to \infty$

Where $\beta_i = \max \mathrm{Re}(\frac{a_j^{(i)}-1}{\alpha_j^{(i)}})$; $j = 1, \ldots, n^{(i)}$, i = 1, ... , r

$$n_2 = n_3 = \ldots = n_r = 0 \tag{1.25}$$

In the contracted notation, this function can be written in the following way:

$$I[z_1,\ldots,z_r] = I^{0,n_2;\ldots;0,n_r:M^{(r)}}_{p_2,q_2;\ldots;p_r,q_r:N^{(r)}}\left[\begin{array}{c|c} z_1 & P_r : P^{(r)} \\ \vdots & \\ z_r & Q_r : Q^{(r)} \end{array}\right].z_1^{s_1}\ldots z_r^{s_r}ds_1\ldots ds_r \tag{1.26}$$

Where $M^{(r)} = (m',n');\ldots;(m^{(r)},n^{(r)});$ (1.27)

$N^{(r)} = (p',q');\ldots;(p^{(r)},q^{(r)});$ (1.28)

$P_r = (a_{2j};\alpha'_{2j},\alpha''_{2j})_{1,p_2} :\ldots: (a_{rj};\alpha'_{rj},\ldots,\alpha^{(r)}_{rj})_{1,p_r};$ (1.29)

$Q_r = (b_{2j};\beta'_{2j},\beta''_{2j})_{1,q_2} :\ldots: (b_{rj};\beta'_{rj},\ldots,\beta^{(r)}_{rj})_{1,q_r};$ (1.30)

$$P^{(r)} = (a_j^{'}, \alpha_j^{'})_{1,p^{'}}; \ldots ; (a_j^{(r)}, \alpha_j^{(r)})_{1,p^{(r)}}; \tag{1.31}$$

$$Q^{(r)} = (b_j^{'}, \beta_j^{'})_{1,q^{'}}; \ldots ; (b_j^{(r)}, \beta_j^{(r)})_{1,q^{(r)}}; \tag{1.32}$$

And the conditions and notations are similar to those given explicitly with (1.19).

H -FUNCTION OF THREE VARIABLES

When $r = 3$, (1.11) reduces to the H -function of three variables defined and represented by means of triple Mellin-Barnes integral in the form:

$$H\begin{bmatrix} x \\ y \\ z \end{bmatrix} = H^{0,n:m_1,n_1;m_2,n_2;m_3,n_3}_{p,q:p_1,q_1;p_2,q_2;p_3,q_3}$$

$$.\left[\begin{matrix} x \\ y \\ z \end{matrix}\middle| \begin{matrix} (a_j; A_j^{'}, A_j^{''}, A_j^{'''})_{1,p} : (c_j^{'}, C_j^{'})_{1,p_1}; (c_j^{''}, C_j^{''})_{1,p_2}; (c_j^{'''}, C_j^{'''})_{1,p_3} \\ (b_j; B_j^{'}, B_j^{''}, B_j^{'''})_{1,q} : (d_j^{'}, D_j^{'})_{1,q_1}; (d_j^{''}, D_j^{''})_{1,q_2}; (d_j^{'''}, D_j^{'''})_{1,q_3} \end{matrix}\right]$$

$$= \frac{1}{(2\pi\omega)^3} \int_{L_1}\int_{L_2}\int_{L_3} \Phi(s_1, s_2, s_3)\theta_1(s_1)\theta_2(s_2)\theta_3(s_3) \ .x^{s_1} y^{s_2} z^{s_3} ds_1 ds_2 ds_3 \tag{1.33}$$

Where $\omega = \sqrt{-1}$;

$$\Phi(s_1, s_2, s_3) = \frac{\prod_{j=1}^{n} \Gamma(1 - a_j + \sum_{i=1}^{3} A_j^{(i)} s_i)}{\prod_{j=n+1}^{p} \Gamma(a_j - \sum_{i=1}^{3} A_j^{(i)} s_i) \prod_{j=1}^{q} \Gamma(1 - b_j + \sum_{i=1}^{3} B_j^{(i)} s_i)}; \tag{1.34}$$

$$\theta_i(s_i) = \frac{\prod_{j=1}^{m_i} \Gamma(d_j^{(i)} - D_j^{(i)} s_i) \prod_{j=1}^{n_i} \Gamma(1 - \tau_j^{(i)} + C_j^{(i)} s_i)}{\prod_{j=m_i+1}^{q_i} \Gamma(1 - d_j^{(i)} + D_j^{(i)} s_i) \prod_{j=n_i+1}^{p_i} \Gamma(\tau_j^{(i)} - C_j^{(i)} s_i)},$$

$$\forall i \in \{1,\ 2,\ 3\}. \tag{1.35}$$

The conditions of existence of the H -function of three variables can be obtained from (1.14), (1.15) and (1.16) on setting $r = 3$.

H-FUNCTION OF TWO VARIABLES

Mittal and Gupta (1972) defined the H -function of two variables. In the notation of Srivastava and Panda (1976), the H -function of two variables is defined and represented by means of double Mellin-Barnes integral in the form:

$$H\begin{bmatrix} x \\ y \end{bmatrix} = H^{0,n:m_1,n_1;m_2,n_2}_{p,q:p_1,q_1;p_2,q_2} \ .\left[\begin{matrix} x \\ y \end{matrix}\middle| \begin{matrix} (a_j; A_j^{'}, A_j^{''})_{1,p} : (c_j^{'}, C_j^{'})_{1,p_1}; (c_j^{''}, C_j^{''})_{1,p_2} \\ (b_j; B_j^{'}, B_j^{''})_{1,q} : (d_j^{'}, D_j^{'})_{1,q_1}; (d_j^{''}, D_j^{''})_{1,q_2} \end{matrix}\right]$$

$$= -\frac{1}{4\pi^2}\int_{L_1}\int_{L_2}\Phi(s_1,s_2)\theta_1(s_1)\theta_2(s_2)x^{s_1}y^{s_2}ds_1ds_2 \qquad (1.36)$$

The functions $\phi(\xi,\eta)$ and [$\theta_1(\xi)$ and $\theta_2(\eta)$] Can be obtained on setting r = 2 in (1.12) and (1.13) respectively.

We mention below some interesting and useful special cases of the H-function of two variables.

(i) If $A_j^{'} = A_j^{"}(j=1,\ .\ .\ .\ .\ ,p), B_j^{'} = B_j^{"}(j=1,\ .\ .\ .\ .\ ,q)$ in (1.5.1),We obtain the special H-function of two variables studied by a number of workers such as Munot and Kalla(1971), Bora and Kalla(1970), Chaturvedi and Goyal(1972), Saxena, (1971,b),Pathak(1970),Shah(1973),Verna,(1971) and others.

(ii) If we assume all capital letters with their dashes as unity, we obtain a relationship of H-function of two variables and G-function of two variables. $H^{0,n:m_1,n_1;m_2,n_2}_{p,q:p_1,q_1;p_2,q_2}\left[\begin{matrix} x \\ y \end{matrix}\middle| \begin{matrix} (a_j;1,1)_{1,p} : (c_j^{'},1)_{1,p_1};(c_j^{"},)_{1,p_2} \\ (b_j;1,1)_{1,q} : (d_j^{'},1)_{1,q_1};(d_j^{"},1)_{1,q_2} \end{matrix}\right]$

$$= G^{0,n:m_1,n_1;m_2,n_2}_{p,q:p_1,q_1;p_2,q_2}\left[\begin{matrix} x \\ y \end{matrix}\middle| \begin{matrix} (a_p):(c_{p_1}^{'});(c_{p_2}^{"}) \\ (b_q):(d_{q_1}^{'});(d_{q_2}^{"}) \end{matrix}\right] \qquad (1.37)$$

The G-function of two variables appearing on the R.H.S. of (1.5.2) was introduced by Agarwal,(1965). In the notation of Srivastava and Joshi [(1969), p. 471], $G[x,y]$ is represented as follows:

$$G\begin{bmatrix} x \\ y \end{bmatrix} = G^{0,n:m_1,n_1;m_2,n_2}_{p,q:p_1,q_1;p_2,q_2}\left[\begin{matrix} x \\ y \end{matrix}\middle| \begin{matrix} (a_p):(c_{p_1}^{'});(c_{p_2}^{"}) \\ (b_q):(d_{q_1}^{'});(d_{q_2}^{"}) \end{matrix}\right]$$

$$= -\frac{1}{4\pi i}\int_{L_1}\int_{L_2}\Phi(\xi+\eta)\psi_1(\xi)\psi_2(\eta)x^{\xi}y^{\eta}d\xi d\eta\ , \qquad (1.38)$$

Where an empty product is interpreted as unity,

$$\Phi(\rho) = \frac{\prod_{j=1}^{n}\Gamma(1-a_j+\rho)}{\prod_{j=n+1}^{p}\Gamma(a_j-\rho)\prod_{j=1}^{q}\Gamma(1-b_j+\rho)} \qquad (1.39)$$

$$\psi_1(\xi) = \frac{\prod_{j=1}^{m_1}\Gamma(d_j^{'}-\xi)\prod_{j=1}^{n_1}\Gamma(1-\tau_j^{'}+\xi)}{\prod_{j=m_1+1}^{q_1}\Gamma(1-d_j^{'}+\xi)\prod_{j=n_1+1}^{p_1}\Gamma(\tau_j^{'}-\xi)} \qquad (1.40)$$

And with $\psi_2(\eta)$ defined analogously to $\psi_1(\xi)$ in terms of the parameter sets $(\tau_{p_2}^{"})$ and $(d_{q_2}^{"})$. Here x and y are not equal to zero, p_i, q_i, n_i and m_i, p, q, n are non –negative integers such that $p \geq n \geq 0, q \geq 0, p_i \geq n_i \geq 0,\ q_i \geq m_i \geq 0\ ,(i=1,2)$

For the details of this function one can refer to the original papers by Agarwal, (1965).

GENERALIZED KAMPE DE FERIET FUNCTION

When $n = p, z_i = -z_i, m_i = 1, n_i = p_i, q_i = q_i + 1, a_j = 1 - a_j, b_k = 1 - b_k$ $(j = 1, \ldots, p; k = 1, \ldots, q)$, $c_g^{(i)} = 1 - c_g^{(i)}, d_h^{(i)} = 1 - d_h^{(i)} (g = 1, \ldots, p_i; h = 1, \ldots, q_i)$, $\forall i \in \{1, \ldots, r\}$

in multivariable H-function (1.11), an interesting relationship obtained as

$$H^{0,p:1,p_1;1,p_2;\ldots;1,p_r}_{p,q:p_1,q_1+1;p_2,q_2+1;\ldots;p_r,q_r+1}$$

$$\cdot\left[\begin{matrix}-z_1\\ \vdots\\ -z_r\end{matrix}\middle| \begin{matrix}(1-a_j;A_j^{'},\ldots,A_j^{(r)})_{1,p}:(1-c_j^{'},C_j^{'})_{1,p_1};\ldots;(1-c_j^{(r)},C_j^{(r)})_{1,p_r}\\ (1-b_j;B_j^{'},\ldots,B_j^{(r)})_{1,q}:(1-d_j^{'},D_j^{'})_{1,q_1};\ldots;(1-d_j^{(r)},D_j^{(r)})_{1,q_r}\end{matrix}\right]$$

$$= S^{p:p_1;p_2;\ldots;p_r}_{q:q_1;q_2;\ldots;p_r}\cdot\left[\begin{matrix}(a_j;A_j^{'},\ldots,A_j^{(r)})_{1,p}:(c_j^{'},C_j^{'})_{1,p_1};\ldots;(c_j^{(r)},C_j^{(r)})_{1,p_r};\\ (b_j;B_j^{'},\ldots,B_j^{(r)})_{1,q}:(d_j^{'},D_j^{'})_{1,q_1};\ldots;(d_j^{(r)},D_j^{(r)})_{1,q_r};\end{matrix} z_1,\ldots,z_r\right] \qquad (1.41)$$

$$= \quad S[z_{1\ldots}z_r]$$

Where $S[z_{1\ldots}z_r]$ is the generalized Kampe de Feriet function of variables defined and represented as follows:

$$S[z_1,\ldots,z_r] = \sum_{s_1,\ldots,s_r=0}^{\infty} \Lambda(s_1,\ldots,s_r)\prod_{i=1}^{r}\{\theta_i(s_i)\frac{z_i^{s_i}}{(s_i)!}\} \qquad (1.42)$$

Where, for the convenience,

$$\Lambda(s_1,\ldots,s_r) = \frac{\prod_{j=1}^{p}\Gamma(a_j+\sum_{i=1}^{r}A_j^{(i)}s_i)}{\prod_{j=1}^{q}\Gamma(b_j+\sum_{i=1}^{r}B_j^{(i)}s_i)} \qquad (1.43)$$

And $\theta_i(s_i) = \dfrac{\prod_{j=1}^{p_i}\Gamma(\tau_j^{(i)}+C_j^{(i)}s_i)}{\prod_{j=1}^{q_i}\Gamma(d_j^{(i)}+D_j^{(i)}s_i)}$ $\quad \forall i \in \{1, \ldots, r\}$ (1.44)

The r –tuple series given by (1.42) converges absolutely, if

$$1+\sum_{j=1}^{q}B_j^{(i)}+\sum_{j=1}^{q_i}D_j^{(i)}-\sum_{j=1}^{p}A_j^{(i)}-\sum_{j=1}^{p_i}C_j^{(i)}\geq 0, \forall i \in \{1, \ldots, r\} \qquad (1.45)$$

Where each of the equalities holds when the variables are suitably constrained.

If we set $r=2$, then (1.42), reduces to the generalized Kampe de Feriet function of two variables defined and studied by Srivastava and Daust (1969), represented as

$$S[x,y]=\sum_{s_1,s_2=0}^{\infty}\xi(s_1,s_2)\theta_1(s_1)\theta_2(s_2)\frac{z_1^{s_1}z_2^{s_2}}{s_1!s_2!} \tag{1.46}$$

Where $\xi(s_1,s_2)$ and $\theta_i(s_i)$ $(i=1,2)$ can be easily found by (1.43) and (1.44) respectively on setting $r=2$. The double series involved in (1.46) convergent if

$$1+\sum_{j=1}^{q}B_j^{(i)}+\sum_{j=1}^{q_i}D_j^{(i)}-\sum_{j=1}^{p}A_j^{(i)}-\sum_{j=1}^{p_i}C_j^{(i)}\geq 0,\quad (i=1,2) \tag{1.47}$$

Further, if we take all capital letters with dashes in (1.46) equal to unity, it reduces to the Kampe de Feriet function.

FOX'S H-FUNCTION

When $n=p=q=0$, (1.19); the multivariable H-function break up into the product of r Fox's H-function.

$$H_{p,q:p_1,q_1;\ldots;p_r,q_r}^{0,0:m_1,n_1;\ldots;m_r,n_r}\left[\begin{matrix} z_1 \\ \vdots \\ z_r\end{matrix}\left|\begin{matrix} -:(c_j',C_j')_{1,p_1};\ldots;(c_j^{(r)},C_j^{(r)})_{1,p_r} \\ -:(d_j',D_j')_{1,q_1};\ldots;(d_j^{(r)},D_j^{(r)})_{1,q_r}\end{matrix}\right.\right]$$

$$=\prod_{i=1}^{r}\left\{H_{p_i,q_i}^{m_i,n_i}\left[z_i\left|\begin{matrix}(c_j^{(i)},C_j^{(i)})_{1,p_i}\\(d_j^{(i)},D_j^{(i)})_{1,q_i}\end{matrix}\right.\right]\right\} \tag{1.48}$$

Fox (1961) has introduced the H-function in the field of special function while investigating the most generalized Fourier Kernel in one variable. The H-function is defined in terms of Mellin-Branes type integral as

$$H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right.\right]=H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,A_1),\ldots,(a_p,A_p)\\(b_1,B_1),\ldots,(b_q,B_q)\end{matrix}\right.\right]=\frac{1}{2\pi\omega}\int_L \chi(s)z^s ds \tag{1.49}$$

Where $\omega=\sqrt{-1}$;

$$\chi(s)=\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)} \tag{1.50}$$

Where $D=\sum_{j=1}^{n}A_j-\sum_{j=n+1}^{p}A_j+\sum_{j=1}^{m}B_j-\sum_{j=m+1}^{q}B_j>0$ (1.51)

And $\mu = \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geq 0$ (1.52)

A detailed account of the analytic continuation and asymptotic expansion of the H-function has been given by Braaksma (1963).

G- and H-function have found a large number of applications in Mathematical physics and chemistry,biological, sociological and statistical sciences. In this connection, one can refer to the monographs by Mathai and Saxena (1973, 78).

Gupta, K.C. (1965) evaluated some integrals involving Bessel, Whittaker and H-functions. Gupta and Jain (1966) also evaluated an integral of product of two H-functions generalizing Saxena's formula for the integral of product of two G-functions (1960). Goyal (1970) has evaluated some finite integrals involving the H-function. Anandani (1969, a, b) evaluated integrals associated with generalized associated Legendre function and the H- function.

Gupta, (1965), Jain, (1968) and Rathie (1979, 80) evaluated certain integrals involving H-function.Bajpai (1971, 80), Sharma, (1965) and Shah (1969) have given certain series expansions of H-function in terms of orthogonal polynomails and the H-function.

An integral transformation associated with the H-function is defined and stuided by Gupta and Mittal (1970, 71) and Rattan singh (1968,70).

Mathai and Saxena (1966) used the H-function in the study of certain statistical distributions. A detailed account of the applications of H-function in statistical distributions is available from the monograph by Mathai and Saxena (1978).

Nair and Samar (1971) obtained the differential properties of the H-function. Saxena and Kumbhat (1974) defined certain operators of fractional integration associated with H-function. Buschman (1972) derived some relations of contiguity for the H-function.

THE GENERAL TRIPLES HPERGEOMETRIC SERIES $F^{(3)}[x, y, z]$

Following srivastava,[(1967), p.428], a general triple hypergeometric series $F^{(3)}[x, y, z]$ is defined as:

$$F^{(3)}[x,y,z] = F^{(3)}\left[\begin{matrix}(a)::(b);(b');(b''):(c);(c');(c'');\\(e)::(g);(g');(g''):(h);(h');(h'');\end{matrix}\, x,y,z\right]$$

$$= \sum_{m,n,p=0}^{\infty} \Lambda(m,n,p)\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!} \tag{1.53}$$

Where for convenience,

$$\Lambda(m,n,p) = \frac{\prod_{j=1}^{A}(a_j)_{m+n+p}\prod_{j=1}^{B}(b_j)_{m+n}\prod_{j=1}^{B'}(b'_j)_{n+p}\prod_{j=1}^{B''}(b''_j)_{p+m}}{\prod_{j=1}^{E}(e_j)_{m+n+p}\prod_{j=1}^{G}(g_j)_{m+n}\prod_{j=1}^{G'}(g'_j)_{n+p}\prod_{j=1}^{G''}(g''_j)_{p+m}}$$

$$\cdot\frac{\prod_{j=1}^{C}(c_j)_m\prod_{j=1}^{C'}(c'_j)_n\prod_{j=1}^{C''}(c''_j)_p}{\prod_{j=1}^{H}(h_j)_m\prod_{j=1}^{H'}(h'_j)_n\prod_{j=1}^{H''}(h''_j)_p} \qquad (1.54)$$

The triple hypergeometric series $F^{(3)}[x,y,z]$ defined by (1.53), converges absolutely, when

$$1+E+G+G''+H-A-B-B''-C\geq 0;\ 1+E+G+G'+H'-A-B-B'-C'\geq 0;$$
$$1+E+G'+G''+H''-A-B'-B''-C''\geq 0.$$

THE MULTIVARIABLE A-FUNCTION

The multivariable A-function introduced by Gautam et.al. [1986] will be define and represent it in the following manner :

$$A[z_1,\ldots,z_r]=A^{m,n:(m_1,n_1):\ldots:(m_r,n_r)}_{p,q:(p_1,q_1):\ldots:(p_r,q_r)}$$

$$\left[z_1,\ldots,z_r\left|\begin{matrix}(a_j,A'_j,\ldots,A_j^{(r)})_{1,p}:(a'_j,\alpha'_j)_{1,p'}:\ldots:(a_j^{(r)},\alpha_j^{(r)})_{1,p^{(r)}}\\(b_j,B'_j,\ldots,B_j^{(r)})_{1,q}:(b'_j,\beta'_j)_{1,q'}:\ldots:(b_j^{(r)},\beta_j^{(r)})_{1,q^{(r)}}\end{matrix}\right.\right]$$

$$=\frac{1}{(2\pi w)^r}\int_{L_1}\ldots\int_{L_r}\phi_1(s_1)\ldots\phi_r(s_r)\psi(s_1,\ldots,s_r)z_1^{s_1},\ldots,z_r^{s_r}ds_1\ldots ds_r \qquad (1.55)$$

Where $w=\sqrt{(-1)}$

$$\phi_i(s_i)=\frac{\prod_{j=1}^{m_i}\Gamma\left(b_j^{(i)}-\beta_j^{(i)}s_i\right)\prod_{j=1}^{n_i}\Gamma\left(1-a_j^{(i)}+\alpha_j^{(i)}s_i\right)}{\prod_{j=m_i+1}^{q_i}\Gamma\left(1-b_j^{(i)}+\beta_j^{(i)}s_i\right)\prod_{j=n_i+1}^{p_i}\Gamma\left(a_j^{(i)}-\alpha_j^{(i)}s_i\right)}\ \forall i\in(1,2,\ldots,r) \qquad (1.56)$$

$$\psi(s_1,\ldots,s_r)=\frac{\prod_{j=1}^{n}\Gamma\left(1-a_j+\sum_{i=1}^{r}A_j^{(i)}s_i\right)\prod_{j=1}^{m}\Gamma\left(b_j-\sum_{i=1}^{r}B_j^{(i)}s_i\right)}{\prod_{j=n+1}^{p}\Gamma\left(a_j-\sum_{i=1}^{r}A_j^{(i)}s_i\right)\prod_{j=m+1}^{q}\Gamma\left(1-b_{3j}+\sum_{i=1}^{r}B_j^{(i)}s_i\right)} \qquad (1.57)$$

$\alpha_j^{(i)},\beta_j^{(i)},\alpha_j^{(i)},\beta_j^{(i)}(i=1,\ldots,r)$ are positive numbers, $a_j^{(i)},b_j^{(i)},a_j,b_j(i=1,\ldots,r)$ are complex numbers and here $m_i,n_i,p_i,q_i(i=1,\ldots,r)$ are non-negative integers where $0\leq m_i\leq q_i, 0\leq n_i\leq p_i$. Here (i) denotes the numbers of dashes. The contours L_i in the complex s_i-plane is of the Mellin-Barnes type which runs from $-w\infty$ to $+w\infty$ with indentations, if necessary, to ensure that all the poles of $\Gamma\left(b_j^{(i)}-\beta_j^{(i)}s_i\right)(j=1,\ldots,m_i)$ are separated from those of $\Gamma\left(1-a_j+\sum_{i=1}^{r}\alpha_j^{(i)}s_i\right)(j=1,\ldots,n_r)$.

For further details and asymptotic expansion of the A-function one can refer by Gautam et.al. [1986].

In what follows, the multivariable A-function defined by [1986] will be represented in the contracted notation:

$$A_{p,q:(p_1,q_1):...:(p_r,q_r)}^{m,n:(m_1,n_1):...:(m_r,n_r)}\left[z_1,...,z_r\right] \qquad (1.58)$$

Or simply by $A\left[z_1,...z_r\right]$.

If we take $A_j^{'}s, B_j^{'}s, C_j^{'}s$ and $D_j^{'}s$ as real and positive and $m = 0$, the A-function reduces to multivariable H-function of Srivastava and Panda (1976).

We are using the multivariable A-function in the following concise form throughout the text.

$$A[z_1,...,z_r] = A_{p,q:N_r}^{m,n:M_r}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\middle| \begin{matrix} P:P_r^{(r)} \\ Q:Q_r^{(r)} \end{matrix}\right]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}....\int_{L_r}\theta_1(s_1)....\theta_r(s_r)\Phi(s_1,....,s_r)z_1^{s_1}...z_r^{s_r}\,.ds_1...ds_r \qquad (1.62)$$

Where $\omega = \sqrt{-1}$; $M_r = m_1, n_1;...;m_r, n_r;$ $N_r = p_1, q_1;...;p_r, q_r;$

$P = (a_j; A_j^{'},..., A_j^{r})_{1,p};$ $Q = (b_j; B_j^{'},..., B_j^{r})_{1,q};$ $P_r^{(r)} = (c_j^{'}, C_j^{'})_{1,p_1};...;(c_j^{(r)}, C_j^{(r)})_{1,p_r};$

And the definition of the functions $\theta_i(s_i)$ $i = 1, ..., r$; $\Phi(s_1,....,s_r)$ and the condition of existence of the multivariable A-function are the same as mentioned by Gautam and Goyal (1981).

A-FUNCTION

${}_1(a_j, \alpha_j)_n$ Represents the set of n pairs of parameters the A-function was defined by Gautam and Goyal as

$$A_{p,q}^{m,n}\left[x\middle|\begin{matrix}{}_1(a_j,\alpha_j)_p \\ {}_1(b_j,\beta_j)_q\end{matrix}\right] = \frac{1}{2\pi i}\int_L f(s)x^s ds \qquad (1.63)$$

Where
$$f(s) = \frac{\prod_{j=1}^{m}\Gamma(a_j+\alpha_j s)\prod_{j=1}^{n}\Gamma(1-b_j-\beta_j s)}{\prod_{j=m+1}^{p}\Gamma(1-a_j-\alpha_j s)\prod_{j=n+1}^{q}\Gamma(b_j+\beta_j s)} \qquad (1.64)$$

The integral on the right hand side of (1.63) is convergent when $f > 0$ and $|\arg(ux)| < \frac{f\pi}{2}$, where

$$f = \mathrm{Re}(\sum_{j=1}^{m}\alpha_j - \sum_{j=m+1}^{p}\alpha_j + \sum_{j=1}^{n}\beta_j - \sum_{j=n+1}^{q}\beta_j \,, \quad u = \prod_{j=1}^{p}\alpha_j^{\alpha_j}\prod_{j=1}^{q}\beta_j^{-\beta_j} \qquad (1.65)$$

(1.63) reduces to H-function given by Fox the following relation

$$A_{p,q}^{n,m}\left[x\middle|\begin{matrix}{}_1(1-a_j,\alpha_j)_p \\ {}_1(1-b_j,\beta_j)_q\end{matrix}\right] = H_{p,q}^{m,n}\left[x\middle|\begin{matrix}{}_1(a_j,\alpha_j)_p \\ {}_1(b_j,\beta_j)_q\end{matrix}\right] \qquad (1.66)$$

THE $\overline{H}$-FUNCTION

$$\overline{H}_{P,Q}^{M,N}[z] = \overline{H}_{P,Q}^{M,N}\left[z\left|\begin{matrix}(a_j;\alpha_j;A_j)_{1,N},(a_j;\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right] = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\overline{\phi}(\xi)z^{\xi}d\xi \tag{1.67}$$

where $$\overline{\phi}(\xi) = \frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\left\{\Gamma(1-a_j+\alpha_j\xi)\right\}^{A_j}}{\prod_{j=M+1}^{Q}\left\{\Gamma(1-b_j+\beta_j\xi)\right\}^{B_j}\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j\xi)} \tag{1.68}$$

Which contains fractional powers of the gamma functions. Here, and throughout the paper $a_j(j=1,\ldots,p)$ and $b_j(j=1,\ldots,Q)$ are complex parameters, $\alpha_j \geq 0(j=1,\ldots,P), \beta_j \geq 0(j=1,\ldots,Q)$ (not all zero simultaneously) and exponents $A_j(j=1,\ldots,N)$ and $B_j(j=N+1,\ldots,Q)$ can take on non integer values.

The following sufficient condition for the absolute convergence of the defining integral for the $\overline{H}$ -function given by equation (1.67) have been given by (Buschman and Srivastava[1]).

$$\Omega \equiv \sum_{j=1}^{M}|\beta_j| + \sum_{j=1}^{N}|A_j\alpha_j| - \sum_{j=M+1}^{Q}|\beta_j B_j| - \sum_{j=N+1}^{P}|\alpha_j| > 0 \tag{1.69}$$

and $|\arg(z)| < \frac{1}{2}\pi\Omega$ (1.70)

The behavior of the $\overline{H}$ -function for small values of $|z|$ follows easily from a result recently given by (Rathie [1997],p.306,eq.(6.9)). We have

$$\overline{H}_{P,Q}^{M,N}[z] = 0\left(|z|^{\gamma}\right), \gamma = \min_{1\leq j\leq N}\left[\operatorname{Re}\left(b_j/\beta_j\right)\right], |z| \to 0 \tag{1.71}$$

If we take $A_j = 1(j=1,\ldots,N)$, $B_j = 1(j=M+1,\ldots,Q)$ in (1.66), the function $\overline{H}_{P,Q}^{M,N}$ reduces to the Fox's H-function [1961].

FRACTIONAL INTEGRAL OPERATORS

The definition of fractional integral of order α studied by Riemann-Liouville (1832, a, 76) as follows:

$$f_{\alpha}^{+}(a,x) = \frac{1}{\Gamma(\alpha)}\int_{a}^{x} f(t)(x-t)^{\alpha-1}dt \quad \text{(Right hand)} \tag{1.72}$$

$$f_{\alpha}^{-}(x,b) = \frac{1}{\Gamma(\alpha)}\int_{x}^{b} f(t)(t-x)^{\alpha-1}dt \quad \text{(Left hand)} \tag{1.73}$$

Where a ≤ x ≤ b, $\alpha > 0$ and is Γ a gamma function.
Weyl (1917) defined the fractional integral of order α in the form:

$$f_{\alpha}^{+}(-\infty,x) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{x} f(t)(x-t)^{\alpha-1}dt \tag{1.74}$$

$$f_{\alpha}^{-}(x,+\infty) = \frac{1}{\Gamma(\alpha)}\int_{x}^{+\infty} f(t)(t-x)^{\alpha-1}dt \tag{1.75}$$

Erdelyi (1940) defined the fractional integral of order α as

$$I_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt\,,$$

$$I_x^0 f(x) = f(x) \tag{1.76}$$

$$K_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1} f(t)dt,$$

$$K_x^0 f(x) = f(x) \tag{1.77}$$

Kober (1940) defined and studied the following fractional integrals of order α (using Erdelyi's notation):

$$\zeta\left[f(x)\right] = \zeta\left[\alpha,\beta,\gamma;m,\mu,\eta,a;\, f(x)\right] = \frac{\mu x^{-\eta-1}}{\Gamma(1-\alpha)} \int_0^x F(\alpha,\beta+m;\gamma;\frac{ax^{\mu}}{t^{\mu}})t^{\eta} f(t)dt$$

$$I_x^{\eta,\alpha} f(x) = x^{-\eta-\alpha} I_x^{\alpha} x^{\eta} f(x), \quad I_x^{\eta,0} f(x) = f(x) \tag{1.78}$$

$$K_x^{\eta,\alpha} f(x) = x K_x^{\alpha} x^{-\eta-\alpha} f(x), \quad K_x^{\eta,0} f(x) = f(x) \tag{1.79}$$

Saxena,(1967b) introduced and studied the operators associated with a hypergeometric associated with a hypergeometric function in the following form:

$$\zeta\left[f(x)\right] = \zeta\left[\alpha,\beta;\gamma;m;\, f(x)\right]$$

$$= \frac{x^{-\gamma-1}}{\Gamma(1-\alpha)} \int_0^x F(\alpha,\beta+m;\beta;t/x)t^{\gamma} f(t)dt \tag{1.80}$$

$$\Re\left[f(x)\right] = \Re\left[\alpha,\beta;\delta;m;\, f(x)\right]$$

$$= \frac{x^{\delta}}{\Gamma(1-\alpha)} \int_x^{\infty} F(\alpha,\beta+m;\beta;t/x)t^{-\delta-1} f(t)dt \tag{1.81}$$

Where $F(\alpha,\beta;\gamma;x)$ is the ordinary hypergeometric function, and $\alpha,\beta,\gamma,\delta$ are complex parameters, if $m=0$, these operators reduce to the operators due to Kober (1940).

Kalla and Saxena (1969) generalized the operators (1.80) and (1.81) by means of the following equations:

$$\zeta\left[f(x)\right] = \zeta\left[\alpha,\beta,\gamma;m,\mu,\eta,a;\, f(x)\right]$$

$$= \frac{\mu x^{-\eta-1}}{\Gamma(1-\alpha)} \int_0^x F(\alpha,\beta+m;\gamma;\frac{ax^{\mu}}{t^{\mu}})t^{\eta} f(t)dt \tag{1.82}$$

$$\Re\left[f(x)\right] = \Re\left[\alpha,\beta,\gamma;m,\mu,\delta,a;\, f(x)\right]$$

$$= \frac{\mu x^{\delta}}{\Gamma(1-\alpha)} \int_x^{\infty} F(\alpha,\beta+m;\gamma;\frac{ax^{\mu}}{t^{\mu}})t^{-\delta-1} f(t)dt \tag{1.83}$$

Where $\alpha,\beta,\gamma,\eta,\delta$ and a are complex parameters.

Lowndes (1970) generalized the Kober operators as well as Hankel operators and also derived their inverses.

Saxena, (1966) obtained an inversion formula for the Verma transform by the application of fractional integration operators.

Fox (1963) studied the integral transform in the light of the theory of fractional integration.

Fox (1971, 72) has enumerated the application of L and L^{-1} operators defined below, in solving the integral equations.

$$x^{-\alpha-\beta}L^{-1}[t^{-\alpha}\{x^{\beta}f(x)\}] = I[\alpha,\beta;\, f(x)] \tag{1.84}$$

$$x^{1-\alpha-\beta}L^{-1}[t^{-\alpha}\{x^{\beta-1}f(\frac{1}{x})\}],(X = 1/X)$$
$$= R[\alpha,\beta;\, f(x)] \tag{1.85}$$

Saxena and Kumbhat (1973) introduced a generalization of Kober operators, and further in 1973, they introduced two new fractional integration operators associated with generalized H-function, and also derived their important properties. In another paper (1973), they established some theorems connecting L, L^{-1} and fractional integration operators, which is an extension of the work of Fox (1972). Saxena and Modi (1980, 85) defined and studied the multidimensional fractional integration operators associated with hypergeometric functions. Gupta and Garg (1984) studied certain multidimensional fractional integral operators involving a general multivariable function in their karnel and also established a relationship between multidimensional fractional integral operators and multidimensional integral transforms. The operators involving multivariable H-function as kernal were defined by Banerji and Sethi (1978).

This subject has also been enriched by the researches of workers like Love (1967, 70), Saigo (1984), Srivastava et al. (1990), Hardy and Littlewood (1925), Gupta and Rajani (1988, 90), Kalla (1966, 71), Mathur and krishana (1977), Pathak and Pandey (1989), Lowndes (1985) and several others.

A detailed account of this subject can be found explicitly in the various monographs notably by Nishimoto (1982, 84, 87, 91), Ross (1975), Samko and Kilbas (1987), Mcbride and Roach (1985), Oldham and Spanier (1974) and many others.

$\overline{H}$ -FUNCTION OF TWO VARIABLES

The $\overline{H}$ -function of two variables introduced by Singh and Mandia (2013) will be defined and represented in the following manner:

$$\overline{H}[x,y] = \overline{H}\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \overline{H}^{o,n_1:\, m_2,n_2:m_3,n_2}_{p_1,q_1:p_2,q_2;p_2,q_2}\left[\begin{matrix} x \\ y \end{matrix}\middle| \begin{matrix} (a_j,\alpha_j;A_j)_{1,p_1},(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3} \\ (b_j,\beta_j;B_j)_{1,q_1},(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3} \end{matrix}\right]$$

$$= -\frac{1}{4\pi^2}\int_{L_1}\int_{L_2}\phi_1(\xi,\eta)\phi_2(\xi)\phi_3(\eta)x^{\xi}y^{\eta}d\xi d\eta \tag{1.86}$$

Where $$\phi_1(\xi,\eta) = \frac{\prod_{j=1}^{n_1}\Gamma\left(1-a_j+\alpha_j\xi+A_j\eta\right)}{\prod_{j=n_1+1}^{p_1}\Gamma\left(a_j-\alpha_j\xi-A_j\eta\right)\prod_{j=1}^{q_1}\Gamma\left(1-b_j+\beta_j\xi+B_j\eta\right)} \tag{1.87}$$

$$\phi_2(\xi) = \frac{\prod_{j=1}^{n_2}\left\{\Gamma\left(1-c_j+\gamma_j\xi\right)\right\}^{K_j}\prod_{j=1}^{m_2}\Gamma\left(d_j-\delta_j\xi\right)}{\prod_{j=n_2+1}^{p_2}\Gamma\left(c_j-\gamma_j\xi\right)\prod_{j=m_2+1}^{q_2}\left\{\Gamma\left(1-d_j+\delta_j\xi\right)\right\}^{L_j}} \tag{1.88}$$

$$\phi_3(\eta) = \frac{\prod_{j=1}^{n_3}\left\{\Gamma\left(1-e_j+E_j\eta\right)\right\}^{R_j}\prod_{j=1}^{m_3}\Gamma\left(f_j-F_j\eta\right)}{\prod_{j=n_3+1}^{p_3}\Gamma\left(e_j-E_j\eta\right)\prod_{j=m_3+1}^{q_3}\left\{\Gamma\left(1-f_j+F_j\eta\right)\right\}^{S_j}} \tag{1.89}$$

The general class of polynomials is defined by Srivastava and Panda [1976,1976] as:

$$S_{n_1,\dots,n_r}^{m_1,\dots,m_r}(x_1,\dots,x_r)=\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_r=0}^{[n_r/m_r]}\frac{(-n_1)_{m_1k_1}}{k_1!}\cdots\frac{(-n_r)_{m_rk_r}}{k_r!}$$

$$F[n_1,k_1;\dots;n_r,k_r]x_1^{k_1}\dots x_r^{k_r} \qquad (1.90)$$

A generalized matrix transform or M-transform of a function $f(X)$ of a $m\times m$ real symmetric positive definite or strictly negative definite matrix X is defined as follows:

$$M_f(s)=\int_{X>0}|X|^{s-\frac{m+1}{2}}f(X)dX \qquad (X>0) \qquad (1.91)$$

Whenever $M_f(s)$ exists. Also $f(X)$ is assumed to be a symmetric function i.e. $f(BX)=f(XB)=f\left(B^{1/2}XB^{1/2}\right)$ for $B=B'>0$. When $X<0$ replace X by $-X$ in

M -transform.

The Mellin transform of $f(t)$ will be defined by $M[f(t)]$. We write $s=p^{-1}+it$ where p and t are real. If $p\ge 1, f(t)\in L_p(0,\infty)$, then

$$p=1,\ M[f(t)]=\int_0^\infty t^{s-1}f(t)dt \qquad (1.92)$$

$$p>1,\ M[f(t)]=l.i.m.\int_{1/x}^{x}t^{s-1}f(t)dt \qquad (1.93)$$

Where $l.i.m.$ denotes the usual limit in the mean for L_p -spaces.

The generalized multidimensional integral transform T, defined as:

$$T\{f(x);s_1,\dots,s_r\}=\int_0^\infty\dots\int_0^\infty k\left(s_1x_1,\dots,s_rx_r\right)f(x)dx_1\dots dx_r \qquad (1.94)$$

Where $k\left(s_1x_1,\dots,s_rx_r\right)$ is the kernel of the transform T and the multiple integral occurring is assumed to be convergent.

The type-1 Beta integral is defined as

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx=\int_0^1 y^{\beta-1}(1-y)^{\alpha-1}dy=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \qquad (1.95)$$

$\mathrm{Re}(\alpha)>0, \mathrm{Re}(\beta)>0$.

The type-2 Beta integral is defined as

$$\int_0^1 x^{\alpha-1}(1+x)^{-(\alpha+\beta)}dx = \int_0^1 y^{\beta-1}(1+y)^{-(\alpha+\beta)}dy = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \qquad (1.96)$$

$\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0$.

$$\int_a^b (t-a)^{x-1}(b-t)^{y-1}dt = (a-b)^{x+y-1}B(x,y); \mathrm{Re}(x) > 0, \mathrm{Re}(y) > 0, b < a$$

Vandermonde's theorem Mathai and Saxena ([1973], p.110 (4.1.2)):

$${}_2F_1(-n,b,c;1) = \frac{(c-b)_n}{(c)_n}; c \neq 0,1,2,...$$

The following result:

$$(a)_n = (-1)^n \frac{\Gamma(1-a)}{\Gamma(1-a-n)}$$

Appell's Functions of two variables

$$F_1(a,b,b';c;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}$$

$$= \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(c)_m m!} {}_2F_1[a+m,b';c+m;y]x^m \qquad (1.97)$$

$(|x|<1, |y|<1)$

$$F_2(a,b,b';c,c';x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_m(c')_n} \frac{x^m}{m!} \frac{y^n}{n!}$$

$$= \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(c)_m m!} {}_2F_1[a+m,b';c';y]x^m \qquad (1.98)$$

$(|x|+|y|<1)$

$$F_3(a,a',b,b';c;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_m(b)_m(b')_n}{(c)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}$$

$$= \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(c)_m m!} {}_2F_1[a',b';c+m;y]x^m \qquad (1.99)$$

$(|x|<1, |y|<1)$

$$F_4(a,b;c,c';x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_{m+n}}{(c)_m(c')_n}\frac{x^m}{m!}\frac{y^n}{n!}$$

$$=\sum_{m=0}^{\infty}\frac{(a)_m(b)_m}{(c)_m m!}\,{}_2F_1[a+m,b+m;c';y]x^m \qquad (1.100)$$

$$(\sqrt{|x|}+\sqrt{|y|}<1)$$

Confluent Hypergeometric Functions of Two Variables

$$\phi_1(a,b;c;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!} \qquad (1.101)$$

$$(|x|<1)$$

$$\phi_2(b,b';c;x,y)=\sum_{m,n=0}^{\infty}\frac{(b)_m(b')_n}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!} \qquad (1.102)$$

$$\phi_3(b;c;x,y)=\sum_{m,n=0}^{\infty}\frac{(b)_m}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!} \qquad (1.103)$$

$$\psi_1(a,b;c,c';x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m}{(c)_m(c')_n}\frac{x^m}{m!}\frac{y^n}{n!} \qquad (1.104)$$

$$(|x|<1)$$

$$\psi_2(a;c,c';x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}}{(c)_m(c')_n}\frac{x^m}{m!}\frac{y^n}{n!} \qquad (1.105)$$

$$\varphi_1(a,a',b;c;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_m(a')_n(b)_m}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!} \qquad (1.106)$$

$$(|x|<1)$$

$$\varphi_2(a,b;c;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_m(b)_m}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!} \qquad (1.107)$$

$$(|x|<1)$$

Lauricella Functions of n-Variables

$$F_A^{(n)}(a,b_1,...,b_n;c_1,...,c_n;x_1,...,x_n)=$$

$$\sum_{m_1,\dots,m_n=0}^{\infty} \frac{(a)_{m_1+\dots+m_n}(b_1)_{m_1}\dots(b_n)_{m_n}}{(c_1)_{m_1}\dots(c_n)_{m_n}} \frac{x_1^{m_1}}{m_1!}\dots\frac{x_n^{m_n}}{m_n!} \qquad (1.108)$$

$(|x_1|+\dots+|x_n|<1)$

$$F_B^{(n)}(a_1,\dots,a_n,b_1,\dots,b_n;c;x_1,\dots,x_n)=$$

$$\sum_{m_1,\dots,m_n=0}^{\infty} \frac{(a_1)_{m_1}\dots(a_n)_{m_n}(b_1)_{m_1}\dots(b_n)_{m_n}}{(c)_{m_1+\dots+m_n}} \frac{x_1^{m_1}}{m_1!}\dots\frac{x_n^{m_n}}{m_n!} \qquad \mathbf{(1.109)}$$

$(|x_1|,1,\dots,|x_n|<1)$

$$F_C^{(n)}(a,b;c_1,\dots,c_n;x_1,\dots,x_n)=\sum_{m_1,\dots,m_n=0}^{\infty} \frac{(a)_{m_1+\dots+m_n}(b)_{m_1+\dots+m_n}}{(c_1)_{m_1}\dots(c_n)_{m_n}} \frac{x_1^{m_1}}{m_1!}\dots\frac{x_n^{m_n}}{m_n!} \qquad \mathbf{(1.110)}$$

$(\sqrt{|x_1|}+\dots+\sqrt{|x_n|}<1)$

$$F_D^{(n)}(a,b_1,\dots,b_n;c;x_1,\dots,x_n)=\sum_{m_1,\dots,m_n=0}^{\infty} \frac{(a)_{m_1+\dots+m_n}(b_1)_{m_1}\dots(b_n)_{m_n}}{(c)_{m_1+\dots+m_n}} \frac{x_1^{m_1}}{m_1!}\dots\frac{x_n^{m_n}}{m_n!} \qquad \mathbf{(1.111)}$$

$(|x_1|<1,\dots,|x_n|<1)$

Also we have

$$F_D^{(n)}(a,b_1,\dots,b_n;c;x,\dots,x)={}_2F_1(a,b_1+\dots+b_n;c;x) \qquad (1.112)$$

Confluent Hypergeometric Functions of Several Variables

$$\phi_2^{(n)}(b_1,\dots,b_n;c;x_1,\dots,x_n)=\sum_{m_1,\dots,m_n=0}^{\infty} \frac{(b_1)_{m_1}\dots(b_n)_{m_n}}{(c)_{m_1+\dots+m_n}} \frac{x_1^{m_1}}{m_1!}\dots\frac{x_n^{m_n}}{m_n!} \qquad \mathbf{(1.113)}$$

$$\psi_2^{(n)}(a,c_1,\dots,c_n;x_1,\dots,x_n)=\sum_{m_1,\dots,m_n=0}^{\infty} \frac{(a)_{m_1+\dots+m_n}}{(c_1)_{m_1}\dots(c_n)_{m_n}} \frac{x_1^{m_1}}{m_1!}\dots\frac{x_n^{m_n}}{m_n!} \qquad \mathbf{(1.114)}$$

Srivastava (1976) introduced the general class of polynomials (see also Srivastava and Singh (1976a)

$$S_n^m[x]=\sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!}A_{n,k}x^k,\ n=0,1,2,\dots \qquad (1.115)$$

Where m and n are arbitrary integers the coefficients $A_{n,k}(n,k\geq 0)$ are arbitrary constants real or complex.

The generalized Struve's function is defined as (2008):

$$H_{v,y,\mu}^{\lambda,k}(z)=\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{z}{2}\right)^{v+2m+1}}{\Gamma(km+y)\Gamma(v+\lambda m+\mu)} \tag{1.116}$$

Where $\mathrm{Re}(k)>0, \mathrm{Re}(\lambda)>0, \mathrm{Re}(y)>0, \mathrm{Re}(v+\mu)>0.$

When $\mu=\frac{3}{2}$, (1.116) reduces to the generalized Struve's function defined by Singh(1989) as:

$$H_{v,y}^{\lambda,k}(z)=\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{z}{2}\right)^{v+2m+1}}{\Gamma(km+y)\Gamma\left(v+\lambda m+\frac{3}{2}\right)} \tag{1.117}$$

Where $\mathrm{Re}(k)>0, \mathrm{Re}(\lambda)>0, \mathrm{Re}(y)>0, \mathrm{Re}(v+\frac{3}{2})>0.$

Put $k=1, y=\frac{3}{2}=\mu$ in (1.116) to get the generalized Struve's function defined by as (2008):

$$H_{v}^{\lambda}(z)=\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{z}{2}\right)^{v+2m+1}}{\Gamma(m+\frac{3}{2})\Gamma\left(v+\lambda m+\frac{3}{2}\right)} \tag{1.118}$$

When $\lambda=1=k, y=\frac{3}{2}=\mu$, (1.116) reduces to the generalized Struve's function defined by Watson (1961) as:

$$H_{v}(z)=\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{z}{2}\right)^{v+2m+1}}{\Gamma(m+\frac{3}{2})\Gamma\left(v+m+\frac{3}{2}\right)} \tag{1.119}$$

Where $\mathrm{Re}\left(v+\frac{3}{2}\right)>0$.

The Beta function is defined as:

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}; \mathrm{Re}(\alpha)>0, \mathrm{Re}(\beta)>0 \tag{1.120}$$

The well known multiplication formula for Gamma functions :

$$\Gamma(mz)=(2\pi)^{\frac{1-m}{2}}m^{mz-\frac{1}{2}}\Gamma(z)\Gamma\left(z+\frac{1}{m}\right)\ldots\Gamma\left(z+\frac{m-1}{m}\right), m=1,2,\ldots$$

Fujiwara (1966) defined a class of generalized polynomials by means of the following

Rodrigues formula:

$$R_n(x)=\frac{(-1)^n k^n}{n!(x-p)^{\beta}(q-x)^{\alpha}}\frac{d^n}{dx^n}\left[(x-p)^{\beta+n}(q-x)^{\alpha+n}\right], p<x<q \, and\, \alpha>-1, \beta>-1 \tag{1.121}$$

We denote these polynomials by $F_n(\beta,\alpha;x)$ and call them extended Jacobi polynomials .

Thakare (1972) obtained the following form of $R_n(x)=F_n(\beta,\alpha;x)$:

$$F_n(\beta,\alpha;x)=\frac{(-1)^n k^n (q-x)^n (1+\beta)_n}{n!}\,{}_2F_1\left[{}^{-n,-n-\alpha}_{1+\beta};\frac{p-x}{q-x}\right], p<x<q$$

Generalized Hypergeometric Functions Involving Two Variables

Kampe de Feriet Function

$$F^{u:v;p}_{w:t;q}\left[{}^{(a_u):(c_v);(e_p)}_{(b_w):(d_t);(f_q)};x,y\right]=\sum_{m,n=0}^{\infty}\frac{\prod_{j=1}^{u}(a_j)_{m+n}\prod_{j=1}^{v}(c_j)_m\prod_{j=1}^{p}(e_j)_n}{\prod_{j=1}^{w}(b_j)_{m+n}\prod_{j=1}^{t}(d_j)_m\prod_{j=1}^{q}(f_j)_n}\frac{x^m}{m!}\frac{y^n}{n!} \tag{1.122}$$

The above series is absolutely convergent for all values of x and y , if

$u+v<w+t+1$ and $u+p<w+q+1$. Also , if $u+v=w+t+1$, we must have any one of the following sets of conditions:

(i) $u\le w,\ for\ \max\{|x||y|\}<1$, (ii) $u>w,\ for\ |x|^{\frac{1}{u-w}}+|y|^{\frac{1}{u-w}}<1$

The Kampe de feriet function has been generalized by Srivastava and Daoust (1969a). Their general function is defined and represented as follows:

$$S[x,y]=S\left[{}^{x}_{y}\right]=S^{p_1:p_2:p_3}_{q_1:p_2:q_3}\left[\begin{array}{c|c} x & (a_j,\alpha_j;A_j)_{1,p_1},(c_j,\gamma_j)_{1,p_2},(e_j,E_j)_{1,p_3} \\ y & (b_j,\beta_j;B_j)_{1,q_1},(d_j,\delta_j)_{1,q_2},(f_j,F_j)_{1,q_3}\end{array}\right]$$

$$=\sum_{m,n=0}^{\infty}\frac{\prod_{j=1}^{p_1}\Gamma\left(a_j+\alpha_j m+A_j n\right)}{\prod_{j=1}^{q_1}\Gamma\left(b_j+\beta_j m+B_j n\right)}\frac{\prod_{j=1}^{p_2}\Gamma\left(c_j+\gamma_j m\right)\prod_{j=1}^{p_3}\Gamma\left(e_j+E_j n\right)}{\prod_{j=1}^{q_2}\Gamma\left(d_j+\delta_j m\right)\prod_{j=1}^{q_3}\Gamma\left(f_j+F_j n\right)}\frac{x^m}{m!}\frac{y^n}{n!} \tag{1.123}$$

The series given by (1.123) converges absolutely, if

$$1+\sum_{j=1}^{q_1}\beta_j+\sum_{j=1}^{q_2}\delta_j-\sum_{j=1}^{p_1}\alpha_j-\sum_{j=1}^{p_2}\gamma_j\ge 0$$

And $$1+\sum_{j=1}^{q_1}B_j+\sum_{j=1}^{q_2}F_j-\sum_{j=1}^{p_1}A_j-\sum_{j=1}^{p_2}E_j\geq 0$$

The following relations are used in the sequal:

$$H_{1,2}^{1,1}\left[x\left|\begin{matrix}(1-k,1)\\(\frac{1}{2}+m,1),(\frac{1}{2}-m,1)\end{matrix}\right.\right]=\frac{\Gamma\left(\frac{1}{2}+k+m\right)}{\Gamma(2m+1)}e^{-\frac{1}{2}x}M_{k,m}(x), \tag{1.124}$$

$$H_{1,2}^{2,0}\left[x\left|\begin{matrix}\left(\frac{1}{2},1\right)\\(b,1),(-b,1)\end{matrix}\right.\right]=\pi^{-\frac{1}{2}}e^{-\frac{1}{2}x}K_b\left(-\frac{1}{2}x\right), \tag{1.125}$$

$$H_{1,2}^{2,0}\left[x\left|\begin{matrix}(1-k,1)\\(\frac{1}{2}+m,1),(\frac{1}{2}-m,1)\end{matrix}\right.\right]=e^{-\frac{1}{2}x}W_{k,m}(x), \tag{1.126}$$

Sethi et. al. discussed the following fractional integral operators involving H -function of matrix arguments:

$$R[f(X)]=R\left[\begin{matrix}(a_j,\alpha_j)_{1,r},(b_j,\beta_j)_{1,s}\\\sigma,\rho,\gamma\end{matrix};f(X)\right]=\frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)}$$

$$\int\limits_{0<U<X}|U|^{\sigma}|X-U|^{\rho-\frac{m+1}{2}}H_{r,s}^{p,q}\left[\gamma(I-UX^{-1})\left|\begin{matrix}(a_j,\alpha_j)_{1,r}\\(b_j,\beta_j)_{1,s}\end{matrix}\right.\right]f(U)dU$$

$$K[f(X)]=K\left[\begin{matrix}(a_j,\alpha_j)_{1,r},(b_j,\beta_j)_{1,s}\\\delta,\rho,\gamma\end{matrix};f(X)\right]=\frac{|X|^{\delta}}{\Gamma_m(\rho)}$$

$$\int\limits_{U>X}|U|^{-\delta-\rho}|U-X|^{\rho-\frac{m+1}{2}}H_{r,s}^{p,q}\left[\gamma(I-XU^{-1})\left|\begin{matrix}(a_j,\alpha_j)_{1,r}\\(b_j,\beta_j)_{1,s}\end{matrix}\right.\right]f(U)dU$$

Where $f(X)=f(x_{11},...,x_{1m},x_{21},...,x_{mm})$ be a real bounded function of a complex parameter.

The fractional (arbitrary) order integral of the function f of order $\alpha>0$ is defined by

$$I_a^{\alpha}f(t)=\int_a^t\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}f(\tau)d\tau.$$

When $a=0,$ we write $I_a^{\alpha}f(t)=f(t)*\phi_{\alpha}(t),$ where (*) denoted the convolution product $\phi_{\alpha}(t)=\frac{t^{\alpha-1}}{\Gamma(\alpha)},t>0$ and $\phi_{\alpha}(t)=0,t\leq 0$ and $\phi_{\alpha}\rightarrow\delta(t)$ as $\alpha\rightarrow 0$ where $\delta(t)$ is the delta function.

The fractional (arbitrary) order derivative of the function f of order $0 \le \alpha < 1$ is defined by

$$D_a^{\alpha} f(t) = \frac{d}{dt}\int_a^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} f(\tau)d\tau$$

$$= \frac{d}{dt} I_a^{1-\alpha} f(t).$$

From above Definitions , we have

$$D^{\alpha}t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} t^{\mu-\alpha},$$

$$\mu > -1; 0 < \alpha < 1$$

and

$$I^{\alpha}t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} t^{\mu+\alpha},$$

$$\mu > -1; \alpha > 0.$$

The goal of this work is to find the exact solution for different kind of fractional differential equations, in terms of H-functions. We consider the non-linear fractional differential equation

$$D_0^{\alpha} u(t) = f(t, u(t)),$$

where $0 < \alpha < 1$, subject to the initial values

$$[D_0^{\alpha-1}u(t)]_{t=0} = [D_0^{-\beta}u(t)]_{t=0}$$

$$= [I_0^{\beta}u(t)]_{t=0} = 0,\ \beta = 1-\alpha.$$

Where $f(t,u(t)) : [0,T] \times \mathrm{R} \to \mathrm{R}$ is a continuous function. Also the exact solution for the linear case

$$D^{\alpha}u(t) - \lambda u(t) = f(t).$$

Furthermore, The multi-term fractional differential equation

$$D^{\alpha}u(t) = f(t, u, D^{\alpha_1}u(t), \ldots, D^{\alpha_n}u(t))$$

The function $F(s)$ on the complex variable s defined by

$$F(s) = \angle\{f(t); s\} = \int_0^{\infty} e^{-st} f(t)dt$$

is called the Laplace transform of the function $f(t)$.

The Mellin transform of the function $f(t)$ is

$$M\{f(t)\}(s) = \int_0^{\infty} t^{s-1} f(t)dt.$$

The Fourier transformation for one dimension is defined as

$$F\{f(r)\}(q) = \int_{R^d} e^{iqr} f(r)dr.$$

The fractional derivative of order α is defined, for a function $f(z)$ by

$$D_z^\alpha f(z) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\alpha} d\zeta,$$

where the function $f(z)$ is analytic in simply-connected region of the complex z-plane C containing the origin and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

The fractional integral of order $\alpha > 0$ is defined, for a function $f(z)$, by

$$I_z^\alpha f(z) := \frac{1}{\Gamma(\alpha)} \int_0^z f(\zeta)(z-\zeta)^{\alpha-1} d\zeta; \alpha > 0,$$

where the function $f(z)$ is analytic in simply-connected region of the complex z-plane (C) containing the origin and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

CHAPTER 2

ON THE DERIVATIVES , PARTIAL DERIVATIVES ,DOUBLE INTEGRAL TRANSFORMATION AND FRACTIONAL INTEGRAL OPERATORS OF A CERTAIN GENERALIZED HYPERGEOMETRIC FUNCTIONS OF ONE AND SEVERAL VARIABLES

In this chapter, methods involving derivatives and the Mellin transformation are employed in obtaining finite summations for the $\overline{H}$ -function of two variables and certain special partial derivatives for the $\overline{H}$ -function of two variables with respect to parameters.

Introduction

The $\overline{H}$ -function of two variables defined and represented by Singh and Mandia (2013) in the following manner:

$$\overline{H}[x,y]=\overline{H}\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]=\overline{H}^{o,n_1:\ m_2,n_2:m_3,n_2}_{p_1,q_1:p_2,q_2;p_2,q_2}\left[\begin{matrix}x\\y\end{matrix}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,p_1},(c_j,C_j;K_j)_{1,n_2},(c_j,C_j)_{n_2+1,p_2},(e_j,\gamma_j;R_j)_{1,n_3},(e_j,\gamma_j)_{n_3+1,p_3}\\(b_j,\beta_j;B_j)_{1,q_1},(d_j,D_j)_{1,m_2},(d_j,D_j;L_j)_{m_2+1,q_2},(f_j,\delta_j)_{1,m_3},(f_j,\delta_j;S_j)_{m_3+1,q_3}\end{matrix}\right.\right]$$

$$=-\frac{1}{4\pi^2}\int_{L_1}\int_{L_2}\phi(\xi,\eta)\theta(\xi)\psi(\eta)x^{\xi}y^{\eta}d\xi d\eta \qquad (2.1.1)$$

Where

$$\phi(\xi,\eta)=\frac{\prod_{j=1}^{n_1}\Gamma\left(1-a_j+\alpha_j\xi+A_j\eta\right)}{\prod_{j=n_1+1}^{p_1}\Gamma\left(a_j-\alpha_j\xi-A_j\eta\right)\prod_{j=1}^{q_1}\Gamma\left(1-b_j+\beta_j\xi+B_j\eta\right)} \qquad (2.1.2)$$

$$\theta(\xi)=\frac{\prod_{j=1}^{n_2}\left\{\Gamma\left(1-c_j+C_j\xi\right)\right\}^{K_j}\prod_{j=1}^{m_2}\Gamma\left(d_j-D_j\xi\right)}{\prod_{j=n_2+1}^{p_2}\Gamma\left(c_j-C_j\xi\right)\prod_{j=m_2+1}^{q_2}\left\{\Gamma\left(1-d_j+D_j\xi\right)\right\}^{L_j}} \qquad (2.1.3)$$

$$\psi(\eta)=\frac{\prod_{j=1}^{n_3}\left\{\Gamma\left(1-e_j+\gamma_j\eta\right)\right\}^{R_j}\prod_{j=1}^{m_3}\Gamma\left(f_j-\delta_j\eta\right)}{\prod_{j=n_3+1}^{p_3}\Gamma\left(e_j-\gamma_j\eta\right)\prod_{j=m_3+1}^{q_3}\left\{\Gamma\left(1-f_j+\delta_j\eta\right)\right\}^{S_j}} \qquad (2.1.4)$$

Where x and y are not equal to zero (complex or real), and an empty product is

interpreted as unity p_i,q_i,n_i,m_j are non-negative integers such that

$0\le n_i\le p_i, o\le m_j\le q_j (i=1,2,3;\ j=2,3)$. All the

$a_j(j=1,2,\ldots,p_1), b_j(j=1,2,\ldots,q_1), c_j(j=1,2,\ldots,p_2), d_j(j=1,2,\ldots,q_2)$,

$e_j(j=1,2,\ldots,p_3)$, $f_j(j=1,2,\ldots,q_3)$ are complex

parameters. $C_j \geq 0 (j=1,2,\ldots,p_2), D_j \geq 0 (j=1,2,\ldots,q_2)$ (not all zero simultaneously),

similarly $\gamma_j \geq 0 (j=1,2,\ldots,p_3), \delta_j \geq 0 (j=1,2,\ldots,q_3)$ (not all zero simultaneously). The

exponents

$K_j(j=1,2,\ldots,n_3), L_j(j=m_2+1,\ldots,q_2), R_j(j=1,2,\ldots,n_3), S_j(j=m_3+1,\ldots,q_3)$

can take on non-negative values.

The contour L_1 is in ξ-plane and runs from $-i\infty$ to $+i\infty$. The poles of

$\Gamma\left(d_j - D_j\xi\right)(j=1,2,\ldots,m_2)$ lie to the right and the poles of

$\Gamma\left\{\left(1-c_j+C_j\xi\right)\right\}^{K_j}(j=1,2,\ldots,n_2), \Gamma\left(1-a_j+\alpha_j\xi+A_j\eta\right)(j=1,2,\ldots,n_1)$ to the left of the

contour. For $K_j(j=1,2,\ldots,n_2)$ not an integer, the poles of gamma functions of the numerator in (2.1.3) are converted to the branch points.

The contour L_2 is in η-plane and runs from $-i\infty$ to $+i\infty$. The poles of

$\Gamma\left(f_j - \delta_j\eta\right)(j=1,2,\ldots,m_3)$ lie to the right and the poles of

$\Gamma\left\{\left(1-e_j+\gamma_j\eta\right)\right\}^{R_j}(j=1,2,\ldots,n_3), \Gamma\left(1-a_j+\alpha_j\xi+A_j\eta\right)(j=1,2,\ldots,n_1)$ to the left of the

contour. For $R_j(j=1,2,\ldots,n_3)$ not an integer, the poles of gamma functions of the numerator in (2.1.4) are converted to the branch points.

The functions defined in (2.1.1) is an analytic function of x and y, if

$$U = \sum_{j=1}^{p_1}\alpha_j + \sum_{j=1}^{p_2}C_j - \sum_{j=1}^{q_1}\beta_j - \sum_{j=1}^{q_2}D_j < 0 \quad (2.1.5)$$

$$V = \sum_{j=1}^{p_1}A_j + \sum_{j=1}^{p_3}\gamma_j - \sum_{j=1}^{q_1}B_j - \sum_{j=1}^{q_3}\delta_j < 0 \quad (2.1.6)$$

The integral in (2.1.1) converges under the following set of conditions:

$$\Delta = \sum_{j=1}^{n_1}\alpha_j - \sum_{j=n_1+1}^{p_1}\alpha_j + \sum_{j=1}^{m_2}D_j - \sum_{j=m_2+1}^{q_2}D_jL_j +$$

$$\sum_{j=1}^{n_2}\gamma_jK_j - \sum_{j=n_2+1}^{p_2}\gamma_j - \sum_{j=1}^{q_1}\beta_j > 0 \quad (2.1.7)$$

$$\Omega=\sum_{j=1}^{n_1}A_j-\sum_{j=n_1+1}^{p_1}A_j+\sum_{j=1}^{m_2}\delta_j-\sum_{j=m_2+1}^{q_2}\delta_jS_j+$$

$$\sum_{j=1}^{n_3}E_jR_j-\sum_{j=n_2+1}^{p_3}E_j-\sum_{j=1}^{q_1}B_j>0 \tag{2.1.8}$$

$$|\arg x|<\frac{1}{2}\Delta\pi,\ |\arg y|<\frac{1}{2}\Omega\pi \tag{2.1.9}$$

The behavior of the $\overline{H}$ -function of two variables for small values of $|z|$ follows as:

$$\overline{H}[x,y]=0(|x|^{\alpha}|y|^{\beta}),\max\{|x|,|y|\}\to 0 \tag{2.1.10}$$

Where $\alpha=\min\limits_{1\le j\le m_2}\left[\operatorname{Re}\left(\frac{d_j}{D_j}\right)\right]$ $\qquad \beta=\min\limits_{1\le j\le m_2}\left[\operatorname{Re}\left(\frac{f_j}{\delta_j}\right)\right]$ (2.1.11)

For large value of $|z|$,

$$\overline{H}[x,y]=0\{|x|^{\alpha'},|y|^{\beta'}\},\min\{|x|,|y|\}\to 0 \tag{2.1.12}$$

Where

$$\alpha'=\max_{1\le j\le n_2}\operatorname{Re}\left(K_j\frac{c_j-1}{C_j}\right),\ \beta'=\max_{1\le j\le n_3}\operatorname{Re}\left(R_j\frac{e_j-1}{\gamma_j}\right) \tag{2.1.13}$$

Provided that $U<0$ and $V<0$.If we take $K_j=1(j=1,2,...,n_2),L_j=1(j=m_2+1,...,q_2)$,

$R_j=1(j=1,2,...,n_3),S_j=1(j=m_3+1,...,q_3)$

in (2.1.1), the $\overline{H}$ -function of two variables reduces to H -function of two variables due to (1961).

If we set $n_1=p_1=q_1=0$, the $\overline{H}$ -function of two variables breaks up into a product of two $\overline{H}$ -function of one variable namely

$$\overline{H}{}^{0,0:m_2,n_2:m_3,n_3}_{0,0:p_2,q_2:p_3,q_3}\left[\begin{matrix}x\\y\end{matrix}\left|\begin{matrix}\text{—}:(c_j,C_j;K_j)_{1,n_2},(c_j,C_j)_{n+1,p_2}:(e_j,\gamma_j;R_j)_{1,n_3},(e_j,\gamma_j)_{n_3+1,p_3}\\ \text{—}:(D_j,\delta_j)_{1,m_2},(d_j,D_j;L_j)_{m_2+1,q_2}:(f_j,\delta_j)_{1,m_3},(f_j,\delta_j;S_j)_{m_3+1,q_3}\end{matrix}\right.\right]$$

$$=\overline{H}{}^{m_2,n_2}_{p_2,q_2}\left[x\left|\begin{matrix}(c_j,C_j;K_j)_{1,n_2},(c_j,C_j)_{n_2+1,p_2}\\(d_j,D_j)_{1,m_2},(d_j,D_j;L_j)_{m_2+1,q_2}\end{matrix}\right.\right]\overline{H}{}^{m_3,n_3}_{p_3,q_3}\left[y\left|\begin{matrix}(e_j,\gamma_j;R_j)_{1,n_3},(e_j,\gamma_j)_{n_3+1,p_3}\\(f_j,\delta_j)_{1,m_3},(f_j,\delta_j;S_j)_{m_3+1,q_3}\end{matrix}\right.\right] \tag{2.1.14}$$

If $\lambda>0$, we then obtain

$$\lambda^2\overline{H}{}^{0,n_1:\,m_2,n_2:m_3,n_3}_{p_1,q_1:p_2,q_2:p_3,q_3}\left[\begin{matrix}x^{\lambda}\\y^{\lambda}\end{matrix}\left|\begin{matrix}(a_j,\lambda\alpha_j;A_j)_{1,p_1},(c_j,\lambda C_j;K_j)_{1,n_2},(c_j,C\gamma_j)_{n_2+1,p_2},(e_j,\lambda\gamma_j;R_j)_{1,n_3},(e_j,\lambda\gamma_j)_{n_3+1,p_3}\\(b_j,\lambda\beta_j;B_j)_{1,q_1},(d_j,\lambda D_j)_{1,m_2},(d_j,\lambda D_j;L_j)_{m_2+1,q_2},(f_j,\lambda\delta_j)_{1,m_3},(f_j,\lambda\delta_j;S_j)_{m_3+1,q_3}\end{matrix}\right.\right]$$

$$= \overline{H}{}^{0,n_1:\, m_2,n_2:m_3,n_3}_{p_1,q_1:p_2,q_2:p_3,q_3}\left[\begin{matrix} x \\ y \end{matrix}\left|\begin{matrix} (a_j,\alpha_j;A_j)_{1,p_1},(c_j,C_j;K_j)_{1,n_2},(c_j,C_j)_{n_2+1,p_2},(e_j,\gamma_j;R_j)_{1,n_3},(e_j,\gamma_j)_{n_3+1,p_3} \\ (b_j,\beta_j;B_j)_{1,q_1},(d_j,D_j)_{1,m_2},(d_j,D_j;L_j)_{m_2+1,q_2},(f_j,\delta_j)_{1,m_3},(f_j,\delta_j;S_j)_{m_3+1,q_3} \end{matrix}\right.\right] \tag{2.1.15}$$

$$\overline{H}{}^{0,n_1:\, m_2,n_2:m_3,n_3}_{p_1,q_1:p_2,q_2:p_3,q_3}\left[\begin{matrix} 1/x \\ 1/y \end{matrix}\left|\begin{matrix} (a_j,\alpha_j;A_j)_{1,p_1},(c_j,C_j;K_j)_{1,n_2},(c_j,C_j)_{n_2+1,p_2},(e_j,\gamma_j;R_j)_{1,n_3},(e_j,\gamma_j)_{n_3+1,p_3} \\ (b_j,\beta_j;B_j)_{1,q_1},(d_j,D_j)_{1,m_2},(d_j,D_j;L_j)_{m_2+1,q_2},(f_j,\delta_j)_{1,m_3},(f_j,\delta_j;S_j)_{m_3+1,q_3} \end{matrix}\right.\right]$$

$$= \overline{H}{}^{0,n_1:\, m_2,n_2:m_3,n_3}_{p_1,q_1:p_2,q_2:p_3,q_3}\left[\begin{matrix} x \\ y \end{matrix}\left|\begin{matrix} (1-b_j,\beta_j;B_j)_{1,q_1},(1-d_j,D_j)_{1,m_2},(1-d_j,D_j;L_j)_{m_2+1,q_2},(1-f_j,\delta_j)_{1,m_3},(1-f_j,\delta_j;S_j)_{m_3+1,q_3} \\ (1-a_j,\alpha_j;A_j)_{1,p_1},(1-c_j,C_j;K_j)_{1,n_2},(1-c_j,C_j)_{n_2+1,p_2},(1-e_j,\gamma_j;R_j)_{1,n_3},(1-e_j,\gamma_j)_{n_3+1,p_3} \end{matrix}\right.\right] \tag{2.1.16}$$

In next section the following differentiating formulas are also used:

Formula 1

$$D_x^r\left\{x^\lambda \overline{H}\left[z_1x^{h_1}, z_2x^{h_2}\right]\right\} =$$

$$x^{\lambda-r}\,\overline{H}{}^{0,n_1+1:m_2,n_2:m_3,n_3}_{p_1+1,q_1+1:p_2,q_2:p_3,q_3}\left[\begin{matrix} z_1x^{h1} \\ z_2x^{h_2} \end{matrix}\left|\begin{matrix} (-\lambda,h_1,h_2),C,E \\ B,(r-\lambda,h_1,h_2),D,F \end{matrix}\right.\right] \tag{2.1.17}$$

For $h_1, h_2 > 0$ and $\lambda \in \square$.

Formula 2

$$(xD_x - k_1)\dots(xD_x - k_r)\left\{x^\lambda \overline{H}\left[z_1x^{h_1}, z_2x^{h_2}\right]\right\} =$$

$$x^{\lambda}\,\overline{H}{}^{0,n_1+r:m_2,n_2:m_3,n_3}_{p_1+r,q_1+r:p_2,q_2:p_3,q_3}\left[\begin{matrix} z_1x^{h1} \\ z_2x^{h_2} \end{matrix}\left|\begin{matrix} (k_j-\lambda,h_1,h_2)_{1,r},C,E \\ B,(1+k_j-\lambda,h_1,h_2)_{1,r},D,F \end{matrix}\right.\right] \tag{2.1.18}$$

Where $\lambda, k_j \in \square$ $(j = 1, 2, \dots, k)$ and h_1, h_2 are real and positive.

Formula 3

$$(D_x x - k_1)\dots(D_x x - k_r)\left\{x^\lambda \overline{H}\left[z_1x^{h_1}, z_2x^{h_2}\right]\right\} =$$

$$x^{\lambda}\,\overline{H}{}^{0,n_1+r:m_2,n_2:m_3,n_3}_{p_1+r,q_1+r:p_2,q_2:p_3,q_3}\left[\begin{matrix} z_1x^{h1} \\ z_2x^{h_2} \end{matrix}\left|\begin{matrix} (k_j-\lambda-1,h_1,h_2)_{1,r},C,E \\ B,(k_j-\lambda,h_1,h_2)_{1,r},D,F \end{matrix}\right.\right] \tag{2.1.19}$$

Where $\lambda, k_j \in \square$ $(j = 1, 2, \dots, k)$ and h_1, h_2 are real and positive.

Main Results

If t be an arbitrary parameter and α', α'' be positive real numbers, then it can be verified that

$$D''\left\{\overline{H}\left[z_1t^{\alpha'}, z_2t^{\alpha''}\right]\right\} = t^{e-3}\,\overline{H}{}^{o,n_1+1:\quad m_2,n_2:m_3,n_2}_{p_1+1,q_1+1:p_2,q_2;p_2,q_2}$$

$$\left[\begin{matrix} z_1t^{\alpha'} \\ z_2t^{\alpha''} \end{matrix}\left|\begin{matrix} (1-e+n;\alpha',\alpha''),(a_j,\alpha_j;A_j)_{1,p_1},(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3} \\ (b_j,\beta_j;B_j)_{1,q_1},(1-e;\alpha',\alpha''),(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3} \end{matrix}\right.\right] \tag{2.2.1}$$

And

$$t^{e-1}\overline{H}^{0,0:m_2,n_2:m_3,n_3}_{0,0:p_2,q_2:p_3,q_3}\left[\begin{array}{c} z_1 t^{\alpha'} \\ z_2 t^{\alpha''} \end{array}\middle| \begin{array}{l} —:(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n+1,p_2}:(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3} \\ —:(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2}:(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3} \end{array}\right] =$$

$$t^{\frac{(e-1)}{2}}\overline{H}^{m_2,n_2}_{p_2,q_2}\left[z_1 t^{\alpha'} \middle| \begin{array}{l} (c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2} \\ (d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2} \end{array}\right] \times$$

$$t^{\frac{(e-1)}{2}}\overline{H}^{m_3,n_3}_{p_3,q_3}\left[z_2 t^{\alpha''} \middle| \begin{array}{l} (e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3} \\ (f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3} \end{array}\right] \qquad (2.2.2)$$

Differentiating (2.2.2) two times w.r.t. t and simplifying, it follows by induction that

$$\overline{H}^{0,1:m_2,n_2:m_3,n_3}_{1,1:p_2,q_2:p_3,q_3}\left[\begin{array}{c} z_1 t^{\alpha'} \\ z_2 t^{\alpha''} \end{array}\middle| \begin{array}{l} (1-e+n;\alpha',\alpha''):(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n+1,p_2}:(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3} \\ (1-e;\alpha',\alpha''):(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2}:(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3} \end{array}\right]$$

$$= \sum_{\substack{n_1,n_2=0 \\ n_1+n_2=n}}^{n} \frac{n!}{n_1!\,n_2!}\overline{H}^{0,0:m_2,n_2+1:m_3,n_3+1}_{0,0:p_2+1,q_2+1:p_3+1,q_3+1}$$

$$\left[\begin{array}{c} z_1 t^{\alpha'} \\ z_2 t^{\alpha''} \end{array}\middle| \begin{array}{l} —:\left(1+n_1-\frac{e+1}{2},\alpha';1\right)(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n+1,p_2}:\left(1+n_2-\frac{e+1}{2},\alpha'';1\right)(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3} \\ —:(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},\left(1-\frac{e+1}{2},\alpha';1\right):(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3},\left(1-\frac{e+1}{2},\alpha'';1\right) \end{array}\right] \qquad (2.2.3)$$

(2.2.3)readily admits an extension and we have

$$= \sum_{\substack{n_1,n_2=0 \\ n_1+n_2=n}}^{n} \frac{n!}{n_1!\,n_2!}\overline{H}^{o,n_1:\ m_2,n_2+1:\ \ m_3,n_2+1}_{p_1,q_1:p_2+1,q_2+1;\,p_2+1,q_2+1}$$

$$\left[\begin{array}{c} z_1 t^{\alpha'} \\ z_2 t^{\alpha''} \end{array}\middle| \begin{array}{l} (a_j,\alpha_j;A_j)_{1,p_1},\left(1+n_1-\frac{e+1}{2},\alpha';1\right),(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},\left(1+n_2-\frac{e+1}{2},\alpha'';1\right),(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3} \\ (b_j,\beta_j;B_j)_{1,q_1},(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},\left(1-\frac{e+1}{2},\alpha';1\right),(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3},\left(1-\frac{e+1}{2},\alpha'';1\right) \end{array}\right] \qquad (2.2.4)$$

Considering various other forms that (2.2.1)admits, similar other results can be obtained. In the next place, in view of (2.2.1) we note that the 2-dimensional Mellin-transformation ([1950],11.2) M'' of the $\overline{H}$ -function of two variables is given by

$$M''(\overline{H}) = Q(-\xi,-\eta)$$

Provided $\quad -\min\limits_{1\le j\le m_2} \operatorname{Re}\left(\frac{d_j}{\delta_j}\right) < \xi < \max\limits_{1\le j\le n_2} \operatorname{Re}\left(K_j\,\frac{c_j-1}{\gamma_j}\right)$

$$-\min_{1\le j\le m_3} \operatorname{Re}\left(\frac{f_j}{F_j}\right) < \eta < \max_{1\le j\le n_3} \operatorname{Re}\left(R_j\,\frac{e_j-1}{E_j}\right)$$

We also note that, since (1.7 (30) of Erdelyi [1953]) for a positive integer N ,

$$\psi(a+N)-\psi(a) = \sum_{k=1}^{N}\frac{(-1)^{k-1}N!}{k\,(N-k)!}\frac{\psi(a)}{\psi(a+k)};\psi(a) = \frac{\Gamma'(a)}{\Gamma(a)},$$

Partial differentiation of the gamma product

$\Gamma\left(1-\frac{e}{2}+\alpha'\xi+\alpha''\eta\right)\Gamma\left(1+\frac{e}{2}+\alpha'\xi+\alpha''\eta\right)$ w.r.t. the arbitrary parameter e at

$e = N$ can be expressed as a finite sum

$$\frac{\partial}{\partial e}\left\{\Gamma\left(1-\frac{e}{2}+\alpha'\xi+\alpha''\eta\right)\Gamma\left(1+\frac{e}{2}+\alpha'\xi+\alpha''\eta\right)\right\}\Bigg|_{e=N}$$

$$=\frac{1}{2}\Gamma\left(1-\frac{N}{2}+\alpha'\xi+\alpha''\eta\right)\sum_{k=1}^{N}\frac{(-1)^{k-1}N!}{k(N-k)!}\frac{\Gamma\left(1+\frac{N}{2}+\alpha'\xi+\alpha''\eta\right)}{\Gamma\left(1+\frac{N}{2}-k+\alpha'\xi+\alpha''\eta\right)},$$

Where α',α'' are positive real numbers.

Thus for $n>0, N>0$, we have

$$M''\left\{\overline{H}^{o,n_1+2:\,m_2,n_2:m_3,n_2}_{p_1+2,q_1:p_2,q_2;p_2,q_2}\left[\begin{matrix}z_1\\ z_2\end{matrix}\right.\right.$$

$$\left.\left.\left|\begin{matrix}\left(-\frac{e}{2};\alpha',\alpha''\right),\left(\frac{e}{2};\alpha',\alpha''\right),(a_j,\alpha_j;A_j)_{1,p_1},,(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},\left(1+n_2-\frac{e+1}{2},\alpha'';1\right),(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3}\\ (b_j,\beta_j;B_j)_{1,q_1},(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},\left(1-\frac{e+1}{2},\alpha';1\right),(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3},\left(1-\frac{e+1}{2},\alpha'';1\right)\end{matrix}\right.\right]_{e=N}\right\}$$

$$=\frac{N!}{2}\sum_{k=1}^{N}\frac{(-1)^{k-1}N!}{k(N-k)!}\frac{\Gamma\left(1+\frac{N}{2}-\alpha'\xi-\alpha''\eta\right)\Gamma\left(1-\frac{N}{2}-\alpha'\xi-\alpha''\eta\right)}{\Gamma\left(1+\frac{N}{2}-k-\alpha'\xi-\alpha''\eta\right)}$$

$$\times Q(-\xi,-\eta)$$

$$M''\left\{\frac{N!}{2}\sum_{k=1}^{N}\frac{(-1)^{k-1}}{k(N-k)!}\overline{H}^{o,n_1+3:\quad m_2,n_2:m_3,n_2}_{p_1+3,q_1+1:p_2,q_2;p_2,q_2}\left[\begin{matrix}z_1\\ z_2\end{matrix}\right.\right.$$

$$\left.\left.\left|\begin{matrix}\left(-\frac{N}{2};\alpha',\alpha''\right),\left(-\frac{N}{2};\alpha',\alpha''\right),\left(\frac{N}{2};\alpha',\alpha''\right),(a_j,\alpha_j;A_j)_{1,p_1},,(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3}\\ (b_j,\beta_j;B_j)_{1,q_1},\left(-\frac{N}{2}+k;\alpha',\alpha''\right),(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3}\end{matrix}\right.\right]\right\} \qquad (2.2.5)$$

But for $z^{(i)}=u^{-\alpha^{(i)}}, i=1,2$, (2.2.5) can be written as

$$\frac{\partial}{\partial e}\left\{\overline{H}^{o,n_1+2:\,m_2,n_2:m_3,n_2}_{p_1+2,q_1:p_2,q_2;p_2,q_2}\left[\begin{matrix}z_1\\ z_2\end{matrix}\right.\right.$$

$$\left.\left.\left|\begin{matrix}\left(-\frac{e}{2};\alpha',\alpha''\right),\left(\frac{e}{2};\alpha',\alpha''\right),(a_j,\alpha_j;A_j)_{1,p_1},,(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3}\\ (b_j,\beta_j;B_j)_{1,q_1},(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3}\end{matrix}\right.\right]_{e=N}\right\}$$

$$=\frac{N!}{2}\sum_{k=1}^{N}\frac{(-1)^{N}u^{\frac{N}{2}}}{k(N-k)!}$$

$$D_u^{N-k}\left\{u^{\frac{N}{2}-k}\,\overline{H}^{o,n_1+2:\ m_2,n_2:m_3,n_2}_{p_1+2,q_1:p_2,q_2;p_2,q_2}\left[\begin{matrix}z_1\\z_2\end{matrix}\right.\right.$$

$$\left.\left.\left|\begin{matrix}\left(-\frac{N}{2};\alpha',\alpha''\right),\left(\frac{N}{2};\alpha',\alpha''\right),(a_j,\alpha_j;A_j)_{1,p_1},,(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3}\\(b_j,\beta_j;B_j)_{1,q_1},(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3}\end{matrix}\right.\right]\right\} \quad (2.2.6)$$

If we express the derivative into a sum, carry out the differentiations, interchange the order of summation and simplify, we obtain

$$\frac{\partial}{\partial e}\left\{\overline{H}^{o,n_1+2:\ m_2,n_2:m_3,n_2}_{p_1+2,q_1:p_2,q_2;p_2,q_2}\left[\begin{matrix}z_1\\z_2\end{matrix}\right.\right.$$

$$\left.\left.\left|\begin{matrix}\left(-\frac{e}{2};\alpha',\alpha''\right),\left(\frac{e}{2};\alpha',\alpha''\right),(a_j,\alpha_j;A_j)_{1,p_1},,(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3}\\(b_j,\beta_j;B_j)_{1,q_1},(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3}\end{matrix}\right.\right]_{e=N}\right\}$$

$$=\frac{N!}{2}\sum_{p=0}^{N-1}\frac{(-1)}{p!(N-p)!}\overline{H}^{o,n_1+2:\ m_2,n_2:m_3,n_2}_{p_1+2,q_1:p_2,q_2;p_2,q_2}$$

$$\left[\begin{matrix}z_1\\z_2\end{matrix}\right.\left|\begin{matrix}\left(-\frac{N}{2};\alpha',\alpha''\right),\left(\frac{N}{2}+p;\alpha',\alpha''\right),(a_j,\alpha_j;A_j)_{1,p_1},,(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3}\\(b_j,\beta_j;B_j)_{1,q_1},(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3}\end{matrix}\right]\Bigg\} \quad (2.2.7)$$

Similar other results can be obtained by considering products or quotients of such gamma functions whose partial derivatives w.r.t. the arbitrary parameter involved can be expressed as a finite sum.For example, for the quotient

$$\frac{\Gamma(1-e-N+\alpha'\xi+\alpha''\eta)}{\Gamma(1-e+\alpha'\xi+\alpha''\eta)},$$

We have

$$\frac{\partial}{\partial e}\left\{\overline{H}^{o,n_1+1:\ \ m_2,n_2:m_3,n_2}_{p_1+1,q_1+1:p_2,q_2;p_2,q_2}\left[\begin{matrix}z_1\\z_2\end{matrix}\right.\right.$$

$$\left.\left.\left|\begin{matrix}(e+N;\alpha',\alpha''),\left(\frac{e}{2};\alpha',\alpha''\right),(a_j,\alpha_j;A_j)_{1,p_1},,(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3}\\(b_j,\beta_j;B_j)_{1,q_1},(e;\alpha',\alpha''),(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3}\end{matrix}\right.\right]\right\}$$

$$=N!\sum_{k=0}^{N}\frac{(-1)^{k-1}}{p(N-k)!}\overline{H}^{o,n_1+1:\ \ m_2,n_2:m_3,n_2}_{p_1+1,q_1+1:p_2,q_2;p_2,q_2}$$

$$\left[\begin{matrix}z_1\\z_2\end{matrix}\right.\left|\begin{matrix}(e+N;\alpha',\alpha''),(a_j,\alpha_j;A_j)_{1,p_1},(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3}\\(b_j,\beta_j;B_j)_{1,q_1},(e+k;\alpha',\alpha''),(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3}\end{matrix}\right]\Bigg\} \quad (2.2.8)$$

We establish the methods involving derivatives and the Mellin transformation are employed in obtaining finite summations for the I -function of two variables and certain special partial derivatives for the I -function of two variables with respect to parameters.

Introduction

The I -function of two variables introduced by Prasad will be represented and defined as follows:

$$I[z_1, z_2] = I^{0,n_2:(m'.n'):(m'',n'')}_{p_2,q_2:(p',q'):(p'',q'')}\left[\begin{array}{c|c} z_1 & (a_{2j}:\alpha'_{2j},\alpha''_{2j})_{1,p_2}:(a'_j,\alpha'_j)_{1,p'}:(a'_j,\alpha'_j)_{1,p''} \\ z_2 & (b_{2j}:\beta'_{2j},\beta''_{2j})_{1,q_2}:(b'_j,\beta'_j)_{1,q'}:(b'_j,\beta'_j)_{1,q''} \end{array}\right]$$

$$= \frac{1}{(2\pi w)^2}\int_{L_1}\int_{L_2}\phi_1(s_1)\phi_2(s_2)\psi(s_1,s_2)z_1^{s_1}z_2^{s_2}ds_1ds_2 \qquad (2.3.1)$$

Where $w = \sqrt{-1}$,

$$\phi_i(s_i) = \frac{\prod_{j=1}^{m^{(i)}}\Gamma\left(b_j^{(i)} - \beta_j^{(i)}s_i\right)\prod_{j=1}^{n^{(i)}}\Gamma\left(1-a_j^{(i)}+\alpha_j^{(i)}s_i\right)}{\prod_{j=m^{(i)}+1}^{q^{(i)}}\Gamma\left(1-b_j^{(i)}+\beta_j^{(i)}s_i\right)\prod_{j=n^{(i)}1}^{p^{(i)}}\Gamma\left(a_j^{(i)}-\alpha_j^{(i)}s_i\right)}\quad \forall i\in\{1,2\} \qquad (2.3.2)$$

$$\psi(s_1,s_2) = \frac{\prod_{j=1}^{n_2}\Gamma\left(1-a_{2j}+\sum_{i=1}^{2}a_{2j}^{(i)}s_i\right)}{\prod_{j=n_2+1}^{p_2}\Gamma\left(a_{2j}-\sum_{i=1}^{2}a_{2j}^{(i)}s_i\right)\prod_{j=1}^{q_2}\Gamma\left(1-b_{2j}+\sum_{i=1}^{2}\beta_{2j}^{i}s_i\right)} \qquad (2.3.3)$$

Main Results

If t be an arbitrary parameter and α',α'' be positive real numbers, then it can be verified that

$$D''\left\{I\left[z_1t^{\alpha'}, z_2t^{\alpha''}\right] = t^{e-3}\right.$$
$$I^{0,n_2+1:(m'.n'):(m'',n'')}_{p_2+1,q_2+1:(p',q'):(p'',q'')}\left[\begin{array}{c|c} z_1t^{\alpha'} & (1-e+n;\alpha',\alpha''),(a_{2j}:\alpha'_{2j},\alpha''_{2j})_{1,p_2}:(a'_j,\alpha'_j)_{1,p'}:(a'_j,\alpha'_j)_{1,p''} \\ z_2t^{\alpha''} & (b_{2j}:\beta'_{2j},\beta''_{2j})_{1,q_2},(1-e;\alpha',\alpha''):(b'_j,\beta'_j)_{1,q'}:(b'_j,\beta'_j)_{1,q''} \end{array}\right] \qquad (2.4.1)$$

And

$$t^{e-1}I^{0,0:m',n':m'',n''}_{0,0:p',q':p'',q''}\left[\begin{array}{c|c} z_1t^{\alpha'} & (a'_j,\alpha'_j)_{1,p'}:(a'_j,\alpha'_j)_{1,p''} \\ z_2t^{\alpha''} & (b'_j,\beta'_j)_{1,q'}:(b'_j,\beta'_j)_{1,q''} \end{array}\right] =$$

$$t^{\frac{(e-1)}{2}}I^{m',n'}_{p',q'}\left[z_1t^{\alpha'}\left|\begin{array}{c}(a'_j,\alpha'_j)_{1,p'}\\(b'_j,\beta'_j)_{1,q'}\end{array}\right.\right]\times t^{\frac{(e-1)}{2}}I^{m'',n''}_{p'',q''}\left[z_2t^{\alpha''}\left|\begin{array}{c}(a'_j,\alpha'_j)_{1,p''}\\(b'_j,\beta'_j)_{1,q''}\end{array}\right.\right] \qquad (2.4.2)$$

Differentiating (2.4.2) two times w.r.t. t and simplifying, it follows by induction that

$$I^{0,1:m',n':m'',n''}_{1,1:p',q':p'',q''}\left[\begin{array}{c|c} z_1t^{\alpha'} & (1-e+n;\alpha',\alpha''),(a'_j,\alpha'_j)_{1,p'},(a'_j,\alpha'_j)_{1,p''} \\ z_2t^{\alpha''} & (1-e;\alpha',\alpha''),(b'_j,\beta'_j)_{1,q'},(b'_j,\beta'_j)_{1,q''} \end{array}\right]$$

$$= \sum_{\substack{n_1,n_2=0\\ n_1+n_2=n}}^{n}\frac{n!}{n_1!n_2!}I^{0,0:m',n'+1:m'',n''+1}_{0,0:p'+1,q'+1:p''+1,q''+1} \qquad (2.4.3)$$
$$\left[\begin{array}{c|c} z_1t^{\alpha'} & \text{---}:\left(1+n_1-\frac{e+1}{2},\alpha'\right),(a'_j,\alpha'_j)_{1,p'},\left(1+n_2-\frac{e+1}{2},\alpha''\right),(a'_j,\alpha'_j)_{1,p''} \\ z_2t^{\alpha''} & \text{---}:(b'_j,\beta'_j)_{1,q'},\left(1-\frac{e+1}{2},\alpha'\right),(b'_j,\beta'_j)_{1,q''},\left(1-\frac{e+1}{2},\alpha''\right) \end{array}\right]$$

(2.4.3)readily admits an extension and we have

$$= \sum_{\substack{n_2,n'=0\\ n_2+n'=n}}^{n} \frac{n!}{n_1!n_2!} I^{o,n_2:\ m',n'+1:\ \ m'',n''+1}_{p_2,q_2:p'+1,q'+1;\,p''+1,q''+1}$$

$$\left[\begin{matrix} z_1 t^{\alpha'} \\ z_2 t^{\alpha''} \end{matrix} \middle| \begin{matrix} (a_{2j}:\alpha'_{2j},\alpha''_{2j})_{1,p_2}, \left(1+n_1-\frac{e+1}{2},\alpha'\right), (a'_j,\alpha'_j)_{1,p'}, \left(1+n_2-\frac{e+1}{2},\alpha''\right), (a''_j,\alpha''_j)_{1,p''} \\ (b_{2j}:\beta'_{2j},\beta''_{2j})_{1,q_2}, (b'_j,\beta'_j)_{1,q'}, \left(1-\frac{e+1}{2},\alpha'\right), (b''_j,\beta''_j)_{1,q''}, \left(1-\frac{e+1}{2},\alpha''\right) \end{matrix} \right] \tag{2.4.4}$$

Considering various other forms that (2.4.1)admits, similar other results can be obtained.

In the next place, in view of (2.4.1) we note that the 2-dimensional Mellin-transformation M'' of the I -function of two variables is given by

$$M''(\overline{H}) = Q(-\xi,-\eta)$$

Provided $\quad -\min_{1\le j\le m'} \operatorname{Re}\left(\frac{b'_j}{\beta'_j}\right) < \xi < \max_{1\le j\le n'} \operatorname{Re}\left(\frac{a'_j-1}{\alpha'_j}\right)$

$$-\min_{1\le j\le m''} \operatorname{Re}\left(\frac{b''_j}{\beta''_j}\right) < \eta < \max_{1\le j\le n''} \operatorname{Re}\left(\frac{a''_j-1}{\alpha''_j}\right)$$

We also note that, since (1.7 (30) of Erdelyi) for a positive integer N ,

$$\psi(a+N)-\psi(a) = \sum_{k=1}^{N} \frac{(-1)^{k-1}N!}{k(N-k)!} \frac{\psi(a)}{\psi(a+k)};\psi(a) = \frac{\Gamma'(a)}{\Gamma(a)},$$

Partial differentiation of the gamma product $\Gamma\left(1-\frac{e}{2}+\alpha'\xi+\alpha''\eta\right)\Gamma\left(1+\frac{e}{2}+\alpha'\xi+\alpha''\eta\right)$ w.r.t. the arbitrary parameter e at $e=N$ can be expressed as a finite sum

$$\frac{\partial}{\partial e}\left\{\Gamma\left(1-\frac{e}{2}+\alpha'\xi+\alpha''\eta\right)\Gamma\left(1+\frac{e}{2}+\alpha'\xi+\alpha''\eta\right)\right\}\Bigg|_{e=N}$$

$$= \frac{1}{2}\Gamma\left(1-\frac{N}{2}+\alpha'\xi+\alpha''\eta\right)\sum_{k=1}^{N}\frac{(-1)^{k-1}N!}{k(N-k)!}\frac{\Gamma\left(1+\frac{N}{2}+\alpha'\xi+\alpha''\eta\right)}{\Gamma\left(1+\frac{N}{2}-k+\alpha'\xi+\alpha''\eta\right)},$$

Where α',α'' are positive real numbers.

Thus for $n>0, N>0$, we have

$$M''\left\{ I^{o,n_2+2:\ m',n':m'',n''}_{p_2+2,q_2:p',q';p'',q''} \left[\begin{matrix} z_1 \\ z_2 \end{matrix} \middle| \begin{matrix} \left(-\frac{e}{2};\alpha',\alpha''\right), \left(\frac{e}{2};\alpha',\alpha''\right), (a_{2j}:\alpha'_{2j},\alpha''_{2j})_{1,p_2} : (a'_j,\alpha'_j)_{1,p'} : (a'_j,\alpha'_j)_{1,p''} \\ (b_{2j}:\beta'_{2j},\beta''_{2j})_{1,q_2}, (b'_j,\beta'_j)_{1,q'} : (b'_j,\beta'_j)_{1,q''} \end{matrix} \right]_{e=N} \right\}$$

$$=\frac{N!}{2}\sum_{k=1}^{N}\frac{(-1)^{k-1}N!}{k(N-k)!}\frac{\Gamma\left(1+\frac{N}{2}-\alpha'\xi-\alpha''\eta\right)\Gamma\left(1-\frac{N}{2}-\alpha'\xi-\alpha''\eta\right)}{\Gamma\left(1+\frac{N}{2}-k-\alpha'\xi-\alpha''\eta\right)}$$

$$\times Q(-\xi,-\eta)$$

$$M''\left\{\frac{N!}{2}\sum_{k=1}^{N}\frac{(-1)^{k-1}}{k(N-k)!}I^{o,n_2+3:\quad m_2,n_2:m_3,n_2}_{p_2+3,q_2+1:p_2,q_2;p_2,q_2}\left[\begin{matrix}z_1\\z_2\end{matrix}\right.\right.$$

$$\left.\left.\left|\begin{matrix}\left(-\frac{N}{2};\alpha',\alpha''\right),\left(-\frac{N}{2};\alpha',\alpha''\right),\left(\frac{N}{2};\alpha',\alpha''\right),(a_{2j};\alpha'_{2j},\alpha''_{2j})_{1,p_2}:(a'_j,\alpha'_j)_{1,p'}:(a'_j,\alpha'_j)_{1,p''}\\(b_{2j};\beta'_{2j},\beta''_{2j})_{1,q_2},\left(-\frac{N}{2}+k;\alpha',\alpha''\right),(b'_j,\beta'_j)_{1,q'}:(b'_j,\beta'_j)_{1,q''}\end{matrix}\right.\right]\right\} \tag{2.4.5}$$

But for $z^{(i)}=u^{-\alpha^{(i)}}, i=1,2$, (2.5) can be written as

$$\frac{\partial}{\partial e}\left\{I^{o,n_2+2:\ m',n':m'',n''}_{p_2+2,q_2:p',q';p'',q''}\left[\begin{matrix}z_1\\z_2\end{matrix}\right.\right.$$

$$\left.\left.\left|\begin{matrix}\left(-\frac{e}{2};\alpha',\alpha''\right),\left(\frac{e}{2};\alpha',\alpha''\right),(a_{2j};\alpha'_{2j},\alpha''_{2j})_{1,p_2}:(a'_j,\alpha'_j)_{1,p'}:(a'_j,\alpha'_j)_{1,p''}\\(b_{2j};\beta'_{2j},\beta''_{2j})_{1,q_2},(b'_j,\beta'_j)_{1,q'}:(b'_j,\beta'_j)_{1,q''}\end{matrix}\right.\right]_{e=N}\right\}$$

$$=\frac{N!}{2}\sum_{k=1}^{N}\frac{(-1)^{N}u^{\frac{N}{2}}}{k(N-k)!}$$

$$D_u^{N-k}\left\{u^{\frac{N}{2}-k}I^{o,n_2+2:\ m',n':m'',n''}_{p_2+2,q_2:p',q';p'',q''}\left[\begin{matrix}z_1\\z_2\end{matrix}\right.\right.$$

$$\left.\left.\left|\begin{matrix}\left(-\frac{N}{2};\alpha',\alpha''\right),\left(\frac{N}{2};\alpha',\alpha''\right),(a_{2j};\alpha'_{2j},\alpha''_{2j})_{1,p_2}:(a'_j,\alpha'_j)_{1,p'}:(a'_j,\alpha'_j)_{1,p''}\\(b_{2j};\beta'_{2j},\beta''_{2j})_{1,q_2},(b'_j,\beta'_j)_{1,q'}:(b'_j,\beta'_j)_{1,q''}\end{matrix}\right.\right]\right\} \tag{2.4.6}$$

If we express the derivative into a sum, carry out the differentiations, interchange the order of summation and simplify, we obtain

$$\frac{\partial}{\partial e}\left\{I^{o,n_2+2:\ m',n':m'',n''}_{p_2+2,q_2:p',q';p'',q''}\left[\begin{matrix}z_1\\z_2\end{matrix}\right.\right.$$

$$\left.\left.\left|\begin{matrix}\left(-\frac{e}{2};\alpha',\alpha''\right),\left(\frac{e}{2};\alpha',\alpha''\right),(a_{2j};\alpha'_{2j},\alpha''_{2j})_{1,p_2}:(a'_j,\alpha'_j)_{1,p'}:(a'_j,\alpha'_j)_{1,p''}\\(b_{2j};\beta'_{2j},\beta''_{2j})_{1,q_2},(b'_j,\beta'_j)_{1,q'}:(b'_j,\beta'_j)_{1,q''}\end{matrix}\right.\right]_{e=N}\right\}$$

$$=\frac{N!}{2}\sum_{p=0}^{N-1}\frac{(-1)}{p!(N-p)!}I^{o,n_2+2:\ m',n':m'',n''}_{p_2+2,q_2:p',q';p'',q''}$$

$$\left[\begin{matrix}z_1\\z_2\end{matrix}\left|\begin{matrix}\left(-\frac{N}{2};\alpha',\alpha''\right),\left(\frac{N}{2}+p;\alpha',\alpha''\right),(a_{2j};\alpha'_{2j},\alpha''_{2j})_{1,p_2}:(a'_j,\alpha'_j)_{1,p'}:(a'_j,\alpha'_j)_{1,p''}\\(b_{2j};\beta'_{2j},\beta''_{2j})_{1,q_2},(b'_j,\beta'_j)_{1,q'}:(b'_j,\beta'_j)_{1,q''}\end{matrix}\right.\right]\right\} \tag{2.4.7}$$

Similar other results can be obtained by considering products or quotients of such gamma functions whose partial derivatives w.r.t. the arbitrary parameter involved can be expressed as a finite sum.

For example, for the quotient

$$\frac{\Gamma(1-e-N+\alpha'\xi+\alpha''\eta)}{\Gamma(1-e+\alpha'\xi+\alpha''\eta)},$$

We have

$$\frac{\partial}{\partial e}\left\{ I^{o,n_2+1:\ m',n':m'',n''}_{p_2+1,q_2+1:p',q';p'',q''}\left[\begin{matrix} z_1 \\ z_2 \end{matrix}\right.\right.$$

$$\left.\left.\left| \begin{matrix} (e+N;\alpha',\alpha''),\left(\frac{e}{2};\alpha',\alpha''\right),(a_{2j}:\alpha'_{2j},\alpha''_{2j})_{1,p_1}:(a'_j,\alpha'_j)_{1,p'}:(a'_j,\alpha'_j)_{1,p''} \\ (b_{2j}:\beta'_{2j},\beta''_{2j})_{1,q_1},(e;\alpha',\alpha''),(b'_j,\beta'_j)_{1,q'}:(b'_j,\beta'_j)_{1,q''} \end{matrix}\right.\right]\right\}$$

$$= N!\sum_{k=0}^{N}\frac{(-1)^{k-1}}{p(N-k)!} I^{o,n_2+1:\ m',n':m'',n''}_{p_2+1,q_2+1:p',q';p'',q''}$$

$$\left[\begin{matrix} z_1 \\ z_2 \end{matrix}\left| \begin{matrix} (e+N;\alpha',\alpha''),(a_{2j}:\alpha'_{2j},\alpha''_{2j})_{1,p_2},(a'_j,\alpha'_j)_{1,p'},(a'_j,\alpha'_j)_{1,p''} \\ (b_{2j}:\beta'_{2j},\beta''_{2j})_{1,q_2},(e+k;\alpha',\alpha''),(b'_j,\beta'_j)_{1,q'},(b'_j,\beta'_j)_{1,q''} \end{matrix}\right.\right]\Bigg\} \qquad (2.4.8)$$

In this chapter, the authors also established a double integral transform of I -function which leads to yet another interesting process of augmenting the parameters in the I -function. The result is of general character and on specializing the parameters suitably, yields several interesting results as particular cases.

Introduction

Rainville ([1960], p.104), Abdul Halim and Al-Salam (1963) have shown that the singleand double Euler transformations of the hypergeometric function ${}_pF_q$ are effective tools for augmenting its parameters. Srivastava and Singhal (1968) and Srivastava and Joshi (1967) have discussed some similar interesting properties of ${}_pF_q$ in double I -function and double Whittaker transforms respectively.

In what follows for the sake of brevity, we have used the symbols

$(a_r,\alpha_r),\Delta(r,a),\Delta(r,\pm a),\Delta((r,a_p))$ to denote the set of parameters

$(a_1,\alpha_1),\ldots,(a_r,\alpha_r);\frac{a}{r},\frac{a+1}{r},\ldots,\frac{a+r-1}{r};\Delta(r,a),\Delta(r,-a)$ and

$\Delta(r,a_1),\Delta(r,a_2),\ldots,\Delta(r,a_p)$ respectively.

The I -function introduced by Saxena (1982) will be represented and defined as follows:

$$I[Z]=I^{m,n}_{p_i,q_i:r}[Z]=I^{m,n}_{p_i,q_i:r}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]=\frac{1}{2\pi\omega}\int_L\theta(s)ds \qquad (2.5.1)$$

where $\omega=\sqrt{-1}$

$$\theta(s) = \frac{\prod_{j=1}^{m} \Gamma(b_j - \beta_j s) \prod_{j=1}^{n} \Gamma(1 - a_j + \alpha_j s)}{\sum_{i=1}^{r} \left\{ \prod_{j=m+1}^{q_i} \Gamma(1 - b_{ji} + \beta_{ji} s) \prod_{j=n+1}^{p_i} \Gamma(a_{ji} - \alpha_{ji} s) \right\}} \qquad (2.5.2)$$

$p_{i,} q_i (i = 1, \ldots, r), m, n$ are integers satisfying $0 \le n \le p_i$, $0 \le m \le q_i, (i = 1, \ldots, r), r$ is

finite $\alpha_j, \beta_j, \alpha_{ij}, \beta_{ji}$ are real and $a_{j,} b_j, a_{ji}, b_{ji}$ are complex numbers such that

$\alpha_j (b_h + v) \ne \beta_h (a_j - v - k)$ for $v, k = 01, 2, \ldots$

We shall use the following notations:

$$A^* = (a_j, \alpha_j)_{1,n}, (a_{ji}, \alpha_{ji})_{n+1, p_i}; B^* = (b_j, \beta_j)_{1,m}, (b_{ji}, \beta_{ji})_{m+1, q_i}$$

$$A^{**} = (c_j, \gamma_j)_{1,r}, (c_{ji}, \gamma_{ji})_{r+1, u_i}; B^{**} = (d_j, \delta_j)_{1,s}, (d_{ji}, \delta_{ji})_{s+1, v_i}$$

If we take $r = 1$ in (2.5.1), the I - function reduces to the Fox's H-function (1961).

Main Result

In this section, we have established the following double integral transform of I -function:

If s, k and r are positive integers, then

$$\int_0^\infty \int_0^\infty x^{\alpha-1} y^{\beta-1} (x+y)^\sigma I_{u_i, v_i : r}^{f,g} \left[\lambda(x+y) \middle| \begin{matrix} C* \\ D* \end{matrix} \right] \; I_{p_i, q_i : r}^{m,n} \left[t x^s y^k (x+y)^r \middle| \begin{matrix} A* \\ B* \end{matrix} \right] dxdy =$$

$$(2\pi)^{(1-D)\left(f+g-\frac{1}{2}u-\frac{1}{2}v\right)+\frac{1}{2}} D^{\sum_1^v d_j - \sum_1^u c_j + \left(A-\frac{1}{2}\right)(u-v)}$$

$$\frac{s^{\alpha-\frac{1}{2}} k^{\beta-\frac{1}{2}}}{\lambda^{\alpha+\beta+\sigma} (s+k)^{\alpha+\beta-\frac{1}{2}}} I_{p_i+\rho+Dv_i, q_i+\rho+Du_i : r}^{m+Dg, n+\rho+Df}$$

$$\left[\frac{t\delta D^{D(v-u)}}{\lambda^D} \middle| \begin{matrix} \Delta((D,1-A-D_f)), \Delta(s,1-\alpha), \Delta(k,1-\beta), A*, \Delta(D,1-A-d_{f+1}), \ldots, \Delta(D,1-A-d_u) \\ \Delta((D,1-A-C_g)), B*, \Delta(k+s,1-\alpha-\beta), \Delta(D,1-A-c_{g+1}), \ldots, \Delta(D,1-A-c_v) \end{matrix} \right] \qquad (2.6.1)$$

Where

$$\gamma = \frac{s^s k^k}{(s+k)^{s+k)}}, \rho = s+k, D = s+k+r, A = \alpha+\beta+\sigma,$$

$$0 \le Dg \le Du \le Dv < Du + q - p, u + v - 2g \le 2f \le 2v, 0 \le n \le p,$$

$$p + q - 2n < 2m \le 2q,$$

$$\operatorname{Re}\left(\min \frac{d_i}{\delta_i} + D \min \frac{b_{ji}}{\beta_{ji}} \right) > \operatorname{Re}(-A) > \operatorname{Re}\left[D\left(\frac{s-\alpha}{s}, \frac{k-\beta}{k}, a_l \right) + C_t - D - 1 \right]$$

$$i=1,2,\ldots,f;\ j=1,2,\ldots,m; l=1,2,\ldots n; t=1,2,\ldots,g; u, \mathrm{Re}(\min C_i+A)-v,$$

$$\mathrm{Re}\left(\max\frac{d_j}{\delta_j}+A\right)-uD+v+\frac{1}{2}D\left(Dv-Du+1\right)>D(Dv-Du),$$

$$\mathrm{Re}\max\left(\frac{s-\alpha}{s},\frac{k-\beta}{k},a_l\right),$$

$$i=1,2,\ldots,u;\ j=1,2,\ldots,v; l=1,2,\ldots,u; |\arg\lambda|\leq\left(f+g-\frac{1}{2}u-\frac{1}{2}v\right)\pi,$$

$$|\arg t|<\left(m+n-\frac{1}{2}p-\frac{1}{2}q\right)\pi, \mathrm{Re}\left(\alpha+s\frac{b_{ji}}{\beta_{ji}}\right)>0, \mathrm{Re}\left(\beta+k\frac{b_{ji}}{\beta_{ji}}\right)>0,\ j=1,2,\ldots,m$$

And the double integral converges.

Proof: To prove (2.6.1), we start with the following known result Erdelyi([1954), p. 177)

$$\int_0^\infty\int_0^\infty \phi(x+y)x^{\alpha-1}y^{\beta-1}dxdy=B(\alpha,\beta)\int_0^\infty\phi(z)z^{\alpha+\beta-1}dz \qquad (2.6.2)$$

Which is valid for $\mathrm{Re}(\alpha)>0$ and $\mathrm{Re}(\beta)>0$.

It is easy to prove by following the technique of reversing the order of integrations, that

$$\int_0^\infty\int_0^\infty \phi(x+y)x^{\alpha-1}y^{\beta-1}I^{m,n}_{p_i,q_i:r}\left[tx^s y^k(x+y)^r\left|{}^{A*}_{B*}\right.\right]dxdy=$$

$$\sqrt{2\pi}\,\frac{s^{\alpha-\frac{1}{2}}k^{\beta-\frac{1}{2}}}{(s+k)^{\alpha+\beta-\frac{1}{2}}}\int_0^\infty\phi(z)z^{\alpha+\beta-1}I^{m,n+\rho}_{p_i+\rho,q_i+\rho:r}\left[t\delta z^D\left|{}^{\Delta(s,1-\alpha),\Delta(k,1-\beta),A*}_{B*,\Delta(k+s,1-\alpha-\beta)}\right.\right]dz \qquad (2.6.3)$$

Where s,k and r are positive integers,

$$\delta=\frac{s^s k^k}{(s+k)^{s+k}}, \rho=s+k, D=s+k+r, p+q<2(m+n),$$

$$|\arg t|<\left(m+n-\frac{1}{2}p-\frac{1}{2}q\right)\pi,$$

$$\mathrm{Re}\left(\alpha+s\frac{b_j}{\beta_j}\right)>0, \mathrm{Re}\left(\beta+k\frac{b_j}{\beta_j}\right)>0,\ j=1,2,\ldots,m.$$

In (2.6.3), taking $\phi(z)=z^\sigma I^{f,g}_{u_i,v_i:r}\left[\lambda z\left|{}^{A**}_{B**}\right.\right]$

And evaluating the integral on the right hand side using ([1960], p.401) the result

(2.6.1) follows.

Particular Cases

On choosing the parameters suitably in (2.6.1), several known and unknown results are obtained as particular cases. However, we mention some of the interesting results here.

(a) Taking

$$f = v = 2, g = 0, u = 1, c_1 = \frac{1}{2}, d_1 = v, d_2 = -v, \sigma = \mu + \frac{1}{2}, \alpha_j = \beta_j = \delta_j = \gamma_j = 1, r = 1$$

in (2.6.1) and using Erdelyi ([1953], p.216, (5))

$$H_{1,2}^{2,0}\left[x\left|\begin{smallmatrix}\left(\frac{1}{2},1\right)\\(b,1),(-b,1)\end{smallmatrix}\right.\right] = \pi^{-\frac{1}{2}} e^{-\frac{1}{2}x} K_b\left(-\frac{1}{2}x\right),$$

We obtain

$$\int_0^\infty\int_0^\infty x^{\alpha-1} y^{\beta-1} (x+y)^{\mu+\frac{1}{2}} a^{-\frac{1}{2}\lambda(x+y)} K_v\left\{-\frac{1}{2}\lambda(x+y)\right\}$$

$$H_{p_i,q_i:1}^{m,n}\left[t x^s y^k (x+y)^r \left|\begin{smallmatrix}A^*\\B^*\end{smallmatrix}\right.\right] dxdy =$$

$$(2\pi)^{-\frac{1}{2}(2-D)\left(f+g-\frac{1}{2}u-\frac{1}{2}v\right)+\frac{1}{2}} D^{A-1}\sqrt{\pi}\,\frac{s^{\alpha-\frac{1}{2}} k^{\beta-\frac{1}{2}}}{\lambda^{\alpha+\beta+\sigma}(s+k)^{\alpha+\beta-\frac{1}{2}}}$$

$$H_{p+\rho+2D,q+\rho+D}^{m,n+\rho+2D}\left[\frac{t\delta D^D}{\lambda^D}\left|\begin{smallmatrix}\Delta((D,1-A\mp v)),\Delta(s,1-\alpha),\Delta(k,1-\beta),A^*\\B^*,\Delta((D,\frac{1}{2}-A)),\Delta(k+s,1-\alpha-\beta)\end{smallmatrix}\right.\right], \qquad (2.7.1)$$

Where δ, D and λ have the same value as (2.6.1) and

$$A = \mu + \alpha + \beta + \frac{1}{2}; p + q < 2(m+n), \text{Re}(\alpha + s\frac{b_{ji}}{\beta_{ji}} \pm v) > 0, \text{Re}(\beta + s\frac{b_{ji}}{\beta_{ji}} \pm v) > 0,$$

$$\text{Re}\left(\alpha + \beta + \mu \pm v + D\frac{b_{ji}}{\beta_{ji}} + \frac{1}{2}\right) > 0, j = 1, 2, \ldots, m; \text{Re}(\lambda) > 0,$$

$$|\arg t| < \left(m + n - \frac{1}{2}p - \frac{1}{2}q\right)\pi$$

(b) Further, replacing q, t and (a_p, α_p) by $q+1, -t$ and $(1-a_p, \alpha_p)$ respectively and then putting $m = 1, n = p, b_1 = 1, b_{j+1} = b_j (j = 1, 2, \ldots q)$, using the result Erdelyi[(1953),p. 215, (1)] and [(1953),p. 4, (11)], we obtain an interesting result obtained by Srivastava and Singhal (1968):

$$\int_0^\infty\int_0^\infty x^{\alpha-1}y^{\beta-1}(x+y)^{\mu+\frac{1}{2}}a^{-\frac{1}{2}\lambda(x+y)}K_v\left\{-\frac{1}{2}\lambda(x+y)\right\}$$

$${}_pF_q\left[{}^{(a_p,\alpha_p)}_{(b_q,\beta_q)}:tx^s y^k(x+y)^r\right]dxdy=$$

$$\frac{\sqrt{\pi}\Gamma\left(\alpha+\beta+\mu\pm v+\frac{1}{2}\right)}{\lambda^{\alpha+\beta+\mu+\frac{1}{2}}\Gamma(\alpha+\beta+\mu+1)}B(\alpha,\beta)$$

$${}_{p+3s+3k+2r}F_{q+2s+2k+r}\left[t\delta\left(\frac{s+k+r}{\lambda}\right)^{s+k+r}\left|\begin{matrix}\Delta(s+k+r,\alpha+\beta+\mu\pm v+\frac{1}{2},),\Delta(s,\alpha),\Delta(k,\beta),(a_p,1)\\(b_q,1),\Delta(s+k,\alpha+\beta),\Delta(k+s,\alpha+\beta+\mu+1)\end{matrix}\right.\right],\qquad(2.7.2)$$

provided $\mathrm{Re}(\mu+\alpha+\beta\pm v+\frac{1}{2})>0,\mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0.$

(c) Setting $v=f=2,g=0,u=0,c_1=\mu,d_1=-\frac{1}{2}-v,d_2=-\frac{1}{2}+v,r=1$ in (2.6.1)

and using the known formula Erdelyi [(1953),p.216,(6)]

$$H_{1,2}^{2,0}\left[x\left|\begin{matrix}(1-k,1)\\(\frac{1}{2}+m,1),(\frac{1}{2}-m,1)\end{matrix}\right.\right]=e^{-\frac{1}{2}x}W_{k,m}(x),$$

We have

$$\int_0^\infty\int_0^\infty x^{\alpha-1}y^{\beta-1}(x+y)^{\sigma}e^{-\frac{1}{2}\lambda(x+y)}W_{\mu,v}[\lambda(x+y)]\,H_{p,q}^{m,n}\left[tx^s y^k(x+y)^r\left|{}^{A^*}_{B^*}\right.\right]dxdy=$$

$$(2\pi)^{\frac{1}{2}(2-D)}D^{\mu+A-\frac{1}{2}}\frac{s^{\alpha-\frac{1}{2}}k^{\beta-\frac{1}{2}}}{\lambda^{\alpha+\beta+\sigma}(s+k)^{\alpha+\beta-\frac{1}{2}}}$$

$$H_{p+\rho+2D,q+\rho+D}^{m,n+\rho+2D}\left[\frac{t\delta D^D}{\lambda^D}\left|\begin{matrix}\Delta(D,\frac{1}{2}-A\pm v),\Delta(s,1-\alpha),\Delta(k,1-\beta),A^*\\B^*,\Delta(k+s,1-\alpha-\beta),\Delta(D,\mu-A)\end{matrix}\right.\right],\qquad(2.7.3)$$

Where D,ρ,δ and A are given in (2.6.1);

$p+q<2(m+n),|\arg t|<\left(m+n-\frac{1}{2}p-\frac{1}{2}q\right)\pi,\mathrm{Re}(\lambda)>0,\mathrm{Re}(\alpha+sb_j>0,$

$\mathrm{Re}(k+sb_j)>0, \mathrm{Re}\left(m+n+\sigma+Db_j\pm v+\frac{1}{2}\right)>0,\ j=1,2,\ldots,m.$

(d) Further, replacing q,t and $\left(a_p,\alpha_p\right)$ by $q+1,-t$ and $(1-a_p,\alpha_p)$ respectively and

then putting $m=1, n=p, b_1=1, b_{j+1}=b_j\,(j=1,2,\ldots,q)$ and using the result

[(1965)p.215,(1)], (3.7.3) reduces to a result due to Srivastava and Joshi

[(1968),p.19,(2.3)]

$$\int_0^\infty\int_0^\infty x^{\alpha-1}y^{\beta-1}(x+y)^{\sigma}e^{-\frac{1}{2}\lambda(x+y)}W_{\mu,v}\left\{\lambda(x+y)\right\}$$

$${}_pF_q\left[{}^{(a_p,\alpha_p)}_{(b_q,\beta_q)}:tx^s y^k(x+y)^r\right]dxdy=\frac{\Gamma\left(\alpha+\beta+\sigma\pm v+\frac{1}{2}\right)}{\lambda^{\alpha+\beta+\sigma}\Gamma\left(\alpha+\beta+\sigma-\mu+1\right)}B(\alpha,\beta)$$

$${}_{p+3s+3k+2r}F_{q+2s+2k+r}\left[t\delta\delta'\left|{}^{\Delta(s+k+r,\alpha+\beta+\sigma\pm v+\frac{1}{2}),\Delta(s,\alpha),\Delta(k,\beta),\left(a_p,1\right)}_{\left(b_q,1\right),\Delta(s+k,\alpha+\beta),\Delta(k+s+r,\alpha+\beta+\sigma-\mu+1)}\right.\right] \tag{2.7.4}$$

Where $\delta=\frac{s^s k^k}{(s+k)^{s+k}}, \delta'=\left(\frac{s+k+r}{\lambda}\right)^{s+k+r}$

$\mathrm{Re}(\alpha)>0, \mathrm{Re}(\beta)>0, \mathrm{Re}(\lambda)>0, \mathrm{Re}\left(\alpha+\beta+\sigma\pm v+\frac{1}{2}\right)>0$ and the resulting hypergeometric series converges.

With $\mu=0, v=\pm\frac{1}{2}$ and $\sigma=-\frac{1}{2}$,(2.7.4) reduces to the earlier results of Jain (1965) and Singh (1965).

(e) Choosing

$$f=g=u=1, v=2, c_1=1-k, d_1=\frac{1}{2}+M, d_2=\frac{1}{2}-M, \alpha_j=\beta_j=\delta_j=\gamma_j=1, r=1$$

in (2.6.1) and using the known result

$$H_{1,2}^{1,1}\left[x\left|{}^{(1-k,1)}_{(\frac{1}{2}+m,1),(\frac{1}{2}-m,1)}\right.\right]=\frac{\Gamma\left(\frac{1}{2}+k+m\right)}{\Gamma(2m+1)}e^{-\frac{1}{2}x}M_{k,m}\left(x\right),$$

We obtain

$$\int_0^\infty\int_0^\infty x^{\alpha-1}y^{\beta-1}(x+y)^{\sigma}e^{-\frac{1}{2}\lambda(x+y)}M_{k,m}[\lambda(x+y)]\,H_{p_i,q_i:1}^{m,n}\left[tx^s y^k(x+y)^r\left|{}^{A*}_{B*}\right.\right]dxdy=$$

$$(2\pi)^{\frac{1}{2}(2-D)} D^{k+A-\frac{1}{2}} \frac{\Gamma(2m+1)}{\Gamma\left(k+m+\frac{1}{2}\right)} \frac{s^{\alpha-\frac{1}{2}} k^{\beta-\frac{1}{2}}}{\lambda^{\alpha+\beta+\sigma}(s+k)^{\alpha+\beta-\frac{1}{2}}}$$

$$H^{m+D,n+\rho+D}_{p+\rho+2D,q+\rho+D}\left[\frac{t\delta D^{D}}{\lambda^{D}}\left|\begin{matrix}\Delta(D,\frac{1}{2}-A-m),\Delta(s,1-\alpha),\Delta(k,1-\beta),\Delta(D,\frac{1}{2}-A+m)\\ \Delta(k+s,1-\alpha-\beta),\Delta(D,k-A)\end{matrix}\right.\right], \qquad (2.7.5)$$

Where D,ρ,δ and A are given in (2.6.1);

$$p+q<2(m+n), |\arg t|<\left(m+n-\frac{1}{2}p-\frac{1}{2}q\right)\pi, \mathrm{Re}(\lambda)>0, \mathrm{Re}(\alpha+sb_j)>0,$$

$$\mathrm{Re}(\beta+kb_j)>0, \mathrm{Re}\left(\alpha+\beta+\sigma+Db_j+m+\frac{1}{2}\right)>0,\ j=1,2,...,m.$$

(f) Substituting $f=1, g=u=0, v=2, d_1=\frac{1}{2}v, d_2=-\frac{1}{2}v, r=1$

and using the result [(1953),p.216,(3)]

$$H^{1,0}_{0,2}\left[x\left|\begin{matrix}-\\ \left(\frac{1}{2}v,1\right),\left(-\frac{1}{2}v,1\right)\end{matrix}\right.\right]=J_v(2\sqrt{x}),$$

(2.6.1) reduces to

$$\int_0^\infty\int_0^\infty x^{\alpha-1}y^{\beta-1}(x+y)^{\sigma}J_v(2\sqrt{\lambda(x+y)})H^{m,n}_{p,q}\left[tx^s y^k\left|\begin{matrix}A^*\\ B^*\end{matrix}\right.\right]dxdy$$

$$=\sqrt{2\pi}\,\frac{D^{2A-1}s^{\alpha-\frac{1}{2}}k^{\beta-\frac{1}{2}}}{\lambda^{\alpha+\beta+\sigma}(s+k)^{\alpha+\beta-\frac{1}{2}}}$$

$$H^{m,n+\rho+D}_{p+\rho+2D,q+p}\left[t\delta\left(\frac{D}{\lambda}\right)^{D}\left|\begin{matrix}\Delta\left(D,1-A-\frac{1}{2}v\right),\Delta(s,1-\alpha),\Delta(k,1-\beta),A^*,\Delta\left(D,1-A+\frac{1}{2}v\right)\\ B^*,\Delta(k+s,1-\alpha-\beta)\end{matrix}\right.\right] \qquad (2.7.6)$$

Where δ, D, ρ and A have the same values given in (2.6.1);

$$p+q<2(m+n), |\arg t|<\left(m+n-\frac{1}{2}p-\frac{1}{2}q\right)\pi, \mathrm{Re}(\lambda)>0, \mathrm{Re}(\alpha+sb_j)>0,$$

$$\mathrm{Re}(\beta+kb_j)>0, \mathrm{Re}\left(\alpha+\beta+\sigma+\frac{1}{2}v+Db_j\right)>0,\ j=1,2,...,m;$$

$$\mathrm{Re}\left(\alpha+\beta+\sigma-D+Da_i\right),\frac{1}{4}, i=1,2,...,n.$$

In view of the numerous properties of I -function, on specializing the parameters suitably, a large number of interesting results may be obtained as particular case.

In the present chapter, the authors introduce two new fractional integration operators associated with $\overline{H}$ -function of two variables. Three important properties of these operators are established which are the generalization of the results given earlier by several authors.

Introduction

The $\overline{H}$ -function of two variables defined and represented by Singh and Mandia (2013) in the following manner:

$$\overline{H}[x,y]=\overline{H}\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]=\overline{H}^{o,n_1:\, m_2,n_2:m_3,n_2}_{p_1,q_1:p_2,q_2;p_2,q_2}\left[\begin{matrix}x\\y\end{matrix}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,p_1},(c_j,C_j;K_j)_{1,n_2},(c_j,C_j)_{n_2+1,p_2},(e_j,\gamma_j;R_j)_{1,n_3},(e_j,\gamma_j)_{n_3+1,p_3}\\(b_j,\beta_j;B_j)_{1,q_1},(d_j,D_j)_{1,m_2},(d_j,D_j;L_j)_{m_2+1,q_2},(f_j,\delta_j)_{1,m_3},(f_j,\delta_j;S_j)_{m_3+1,q_3}\end{matrix}\right.\right]$$

$$=-\frac{1}{4\pi^2}\int_{L_1}\int_{L_2}\phi(\xi,\eta)\theta(\xi)\psi(\eta)x^{\xi}y^{\eta}d\xi d\eta \qquad (2.8.1)$$

Where $$\phi(\xi,\eta)=\frac{\prod_{j=1}^{n_1}\Gamma\left(1-a_j+\alpha_j\xi+A_j\eta\right)}{\prod_{j=n_1+1}^{p_1}\Gamma\left(a_j-\alpha_j\xi-A_j\eta\right)\prod_{j=1}^{q_1}\Gamma\left(1-b_j+\beta_j\xi+B_j\eta\right)} \qquad (2.8.2)$$

$$\theta(\xi)=\frac{\prod_{j=1}^{n_2}\left\{\Gamma\left(1-c_j+C_j\xi\right)\right\}^{K_j}\prod_{j=1}^{m_2}\Gamma\left(d_j-D_j\xi\right)}{\prod_{j=n_2+1}^{p_2}\Gamma\left(c_j-C_j\xi\right)\prod_{j=m_2+1}^{q_2}\left\{\Gamma\left(1-d_j+D_j\xi\right)\right\}^{L_j}} \qquad (2.8.3)$$

$$\psi(\eta)=\frac{\prod_{j=1}^{n_3}\left\{\Gamma\left(1-e_j+\gamma_j\eta\right)\right\}^{R_j}\prod_{j=1}^{m_3}\Gamma\left(f_j-\delta_j\eta\right)}{\prod_{j=n_3+1}^{p_3}\Gamma\left(e_j-\gamma_j\eta\right)\prod_{j=m_3+1}^{q_3}\left\{\Gamma\left(1-f_j+\delta_j\eta\right)\right\}^{S_j}} \qquad (2.8.4)$$

Definitions

We introduce the fractional integration operators by means of the following integral equations:

$$Y\left[\begin{smallmatrix}\delta,\beta;r\\\lambda,\mu;x\end{smallmatrix} f(x)\right]=rx^{-\delta-r\beta-1}\int_0^x t^{\delta}\left(x^r-t^r\right)^{\beta} f(t)\overline{H}\left[\begin{smallmatrix}\lambda U\\\mu U\end{smallmatrix}\right]dt \qquad (2.9.1)$$

And

$$N\left[\begin{smallmatrix}\alpha,\beta;r\\\lambda,\mu;x\end{smallmatrix} f(x)\right]=rx^{\alpha}\int_0^x t^{-\alpha-r\beta-1}\left(t^r-x^r\right)^{\beta} f(t)\overline{H}\left[\begin{smallmatrix}\lambda V\\\mu V\end{smallmatrix}\right]dt \qquad (2.9.2)$$

Where U and V represent the expressions

$$\left(\frac{t^r}{x^r}\right)^m\left(1-\frac{t^r}{x^r}\right)^n \text{ and } \left(\frac{x^r}{t^r}\right)^m\left(1-\frac{x^r}{t^r}\right)^n$$

Respectively, r, m, n are positive integers and

$$|\arg\lambda| < \frac{1}{2}\Omega\pi, \ |\arg\mu| < \frac{1}{2}\Lambda\pi.$$

The conditions of the validity of these operators are as follow:

(i) $1 \le p, q < \infty, \frac{1}{p}+\frac{1}{q}=1$

(ii) $\operatorname{Re}\left(\delta + rm\frac{d_j}{\delta_j} + rm\frac{f_j}{F_j}\right) > -\frac{1}{q}$, $\operatorname{Re}\left(\beta + rn\frac{d_j}{\delta_j} + rn\frac{f_j}{F_j}\right) > -\frac{1}{q}$,

$\operatorname{Re}\left(\alpha + rm\frac{d_j}{\delta_j} + rm\frac{f_j}{F_j}\right) > -\frac{1}{p}$, (iii) $f(x) \in L_p(0,\infty)$.

The last condition ensures that Y and N both exists and also that both belongs to $L_p(0,\infty)$.

If we set $n_1 = p_1 = q_1 = 0$, we obtain the operators involving the product of two $\overline{H}$ -functions.

If we take $K_j = 1(j=1,2,\ldots,n_2), L_j = 1(j = m_2+1,\ldots,q_2), R_j = 1(j=1,2,\ldots,n_3), S_j = 1(j = m_3+1,\ldots,q_3)$ we obtain the operators involving H -function of two variables.

Mellin Transform

The Mellin transform of $f(t)$ will be defined by $M[f(t)]$. We write $s = p^{-1} + it$ where p and t are real. If $p \ge 1, f(t) \in L_p(0,\infty)$, then

$$p=1, \ M[f(t)] = \int_0^\infty t^{s-1} f(t)dt \qquad (2.10.1)$$

$$p>1, \ M[f(t)] = l.i.m.\int_{1/x}^{x} t^{s-1} f(t)dt \qquad (2.10.2)$$

Where $l.i.m.$ denotes the usual limit in the mean for L_p -spaces.

Theorem 1. If

$f(x) \in L_p(0,\infty), 1 \le p \le 2[or\ f(x) \in M_p(0,\infty) and\ p > 2] |\arg\lambda| < \frac{1}{2}\Omega\pi, |\arg\mu| < \frac{1}{2}\Lambda\pi, (\Omega,\Lambda) > 0,$

$$\operatorname{Re}\left(\delta+rm\frac{d_j}{\delta_j}+rm\frac{f_j}{F_j}\right)>-\frac{1}{q},\ \operatorname{Re}\left(\beta+rn\frac{d_j}{\delta_j}+rn\frac{f_j}{F_j}\right)>-\frac{1}{q}, \text{ then}$$

$$M\left\{Y\left[{}^{\delta,\beta;r}_{\lambda,\mu;x} f(x)\right]\right\}=\overline{H}^{o,n_1:\ \ m_2,n_2:m_3,n_2}_{p_1+2,q_1+2:p_2,q_2;p_2,q_2}$$

$$\left[\begin{array}{c|l} x & (a_j,\alpha_j;A_j)_{1,p_1},\left(\frac{-\delta+s}{r},m\right),(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3} \\ y & (b_j,\beta_j;B_j)_{1,q_1},\left(-\beta-\frac{1+\delta-s}{r},m+n\right),(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3}\end{array}\right]M[f(x)] \quad (2.10.3)$$

Where $M_p(0,\infty)$ denotes the class of all functions $f(x)$ of $L_p(0,\infty)$ with $p>2$ which are inverse Mellin transforms of functions of $L_q(-\infty,\infty)$.

Proof: From (2.10.1), it follows that

$$M\left\{Y\left[{}^{\delta,\beta;r}_{\lambda,\mu;x} f(x)\right]\right\}=\int_0^\infty x^{s-1}rx^{-\delta-r\beta-1}\int_0^x t^\delta\left(x^r-t^r\right)^\beta f(t)\overline{H}\left[{}^{\lambda U}_{\mu U}\right]dtdx$$

$$=\int_0^\infty t^\delta f(t)dt\int_0^x x^{s-\delta-r\beta-2}\left(x^r-t^r\right)^\beta f(t)\overline{H}\left[{}^{\lambda U}_{\mu U}\right]dx,$$

The change of order of integration is permissible by virtue of de la Vallee Poisson's theorem (p.504)under the conditions stated with the theorems. The theorem readily follows on evaluating the x-integral by means of the formula

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\overline{H}^{o,n_1:\ m_2,n_2:m_3,n_2}_{p_1,q_1:p_2,q_2;p_2,q_2}\left[\begin{array}{c|l}\lambda x^k(1-x)^l & (a_j,\alpha_j;A_j)_{1,p_1},(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3}\\ \mu x^k(1-x)^l & (b_j,\beta_j;B_j)_{1,q_1},(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3}\end{array}\right]dx$$

$$=\overline{H}^{o,n_1:\ \ m_2,n_2:m_3,n_2}_{p_1+2,q_1+2:p_2,q_2;p_2,q_2}\left[\begin{array}{c|l}\lambda & (a_j,\alpha_j;A_j)_{1,p_1},(1-\alpha,k),(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3}\\ \mu & (b_j,\beta_j;B_j)_{1,q_1},(1-\alpha-\beta,k+l),(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3}\end{array}\right] \quad (2.10.4)$$

Where, $\operatorname{Re}\left(\alpha+k\frac{d_j}{\delta_j}+k\frac{f_j}{F_j}\right)>0, \operatorname{Re}\left(\beta+k\frac{d_j}{\delta_j}+k\frac{f_j}{F_j}\right)>0 \ |\arg\lambda|<\frac{1}{2}\Omega\pi, |\arg\mu|<\frac{1}{2}\Lambda\pi, (\Omega,\Lambda)>0$, which follows from the definition of the $\overline{H}$-function of two variables and Beta function formula.

In a similar manner, the following theorems can be establisded.

Theorem 2. If

$$f(x)\in L_p(0,\infty), 1\le p\le 2[or\ f(x)\in M_p(0,\infty) and\ p>2]\,|\arg\lambda|<\frac{1}{2}\Omega\pi, |\arg\mu|<\frac{1}{2}\Lambda\pi, (\Omega,\Lambda)>0,$$

$$\operatorname{Re}\left(\beta+rn\frac{d_j}{\delta_j}+rn\frac{f_j}{F_j}\right)>-\frac{1}{q},\ \operatorname{Re}\left(\alpha+\beta r+rm\frac{d_j}{\delta_j}+rm\frac{f_j}{F_j}\right)>-\frac{1}{q}, \frac{1}{p}+\frac{1}{q}=1 \text{ then}$$

$$M\left\{N\left[{}^{\alpha,\beta;r}_{\lambda,\mu;x} f(x)\right]\right\}=\overline{H}^{o,n_1:\ \ m_2,n_2:m_3,n_2}_{p_1+2,q_1+2:p_2,q_2;p_2,q_2}$$

$$\left[\begin{matrix} \lambda \\ \mu \end{matrix} \left| \begin{matrix} (a_j,\alpha_j;A_j)_{1,p_1}, \left(\frac{1-\alpha\delta-s}{r},m\right), (c_j,\gamma_j;K_j)_{1,n_2}, (c_j,\gamma_j)_{n_2+1,p_2}, (e_j,E_j;R_j)_{1,n_3}, (e_j,E_j)_{n_3+1,p_3} \\ (b_j,\beta_j;B_j)_{1,q_1}, \left(-\beta-\frac{\alpha+s}{r},m+n\right), (d_j,\delta_j)_{1,m_2}, (d_j,\delta_j;L_j)_{m_2+1,q_2}, (f_j,F_j)_{1,m_3}, (f_j,F_j;S_j)_{m_3+1,q_3} \end{matrix} \right. \right] M[f(x)] \qquad (2.10.5)$$

Theorem 3. If $f(x) \in L_p(0,\infty), g(x) \in L_p(0,\infty), |\arg\lambda| < \frac{1}{2}\Omega\pi, |\arg\mu| < \frac{1}{2}\Lambda\pi, (\Omega,\Lambda) > 0$,

$\mathrm{Re}\left(\beta + rm\frac{d_j}{\delta_j} + rm\frac{f_j}{F_j}\right) > \max\left(\frac{1}{p},\frac{1}{q}\right)$, $\mathrm{Re}\left(\beta r + rm\frac{d_j}{\delta_j} + rm\frac{f_j}{F_j}\right) > 0, \frac{1}{p}+\frac{1}{q} = 1$ then

$$\int_0^\infty g(x) Y\left[\begin{smallmatrix}\delta,\beta;r\\ \lambda,\mu;x\end{smallmatrix} f(x)\right]dx = \int_0^\infty f(x) N\left[\begin{smallmatrix}\delta,\beta;r\\ \lambda,\mu;x\end{smallmatrix} g(x)\right]dx \qquad (2.10.6)$$

Formal Properties of the Operators

Here, we give some formal properties of the operators which follow as consequences of the definitions (2.10.1) and (2.10.2).

$$x^{-1} Y\left[\begin{smallmatrix}\delta,\beta;r\\ \lambda,\mu;x^{-1}\end{smallmatrix} f(x^{-1})\right] = N\left[\begin{smallmatrix}\delta,\beta;r\\ \lambda,\mu;x\end{smallmatrix} f(x)\right] \qquad (2.11.1)$$

$$x^{-1} N\left[\begin{smallmatrix}\alpha,\beta;r\\ \lambda,\mu;x^{-1}\end{smallmatrix} f(x^{-1})\right] = Y\left[\begin{smallmatrix}\alpha,\beta;r\\ \lambda,\mu;x\end{smallmatrix} f(x)\right] \qquad (2.11.2)$$

$$x^{\eta} Y\left[\begin{smallmatrix}\delta,\beta;r\\ \lambda,\mu;x\end{smallmatrix} f(x)\right] = Y\left[\begin{smallmatrix}\delta-\eta,\beta;r\\ \lambda,\mu;x\end{smallmatrix} x^{\eta} f(x)\right] \qquad (2.11.3)$$

$$x^{\eta} N\left[\begin{smallmatrix}\alpha,\beta;r\\ \lambda,\mu;x\end{smallmatrix} f(x)\right] = N\left[\begin{smallmatrix}\alpha+\eta,\beta;r\\ \lambda,\mu;x\end{smallmatrix} x^{\eta} f(x)\right] \qquad (2.11.4)$$

If $Y\left[\begin{smallmatrix}\delta,\beta;r\\ \lambda,\mu;x\end{smallmatrix} f(x)\right] = g(x)$, then

$$Y\left[\begin{smallmatrix}\delta,\beta;r\\ \lambda,\mu;x\end{smallmatrix} f(cx)\right] = g(cx) \qquad (2.11.5)$$

And if $N\left[\begin{smallmatrix}\alpha,\beta;r\\ \lambda,\mu;x\end{smallmatrix} f(x)\right] = \phi(x)$, then

$$N\left[\begin{smallmatrix}\alpha,\beta;r\\ \lambda,\mu;x\end{smallmatrix} f(cx)\right] = \phi(cx) \qquad (2.11.6)$$

In conclusion, it is interesting to observe that double integral operators associated with $\overline{H}$ -function of two variables can be defined in the same way and their various properties, analogous to the integral operators studied in this paper can be obtained.

In the present paper, we study certain multidimensional fractional integral operators involving a general I -function in their kernel. We give five basic properties of these operators, and then establish two theorems and two corollaries, which are believed to be new. These basic theorems exhibit structural relationships between the multidimensional integral transforms. The one-and –two –dimensional analogues of these results, which are new and of interest in themselves, can easily be deduced. Special cases of these latter theorems will give rise to certain known results obtained from time to time by several earlier authors.

Introduction

Fractional integral operators have been defined and studied by various authors notably by Riemann and Liouville , Weyl , Erdelyi , Kober , Sneddon , Kalla , Saxena , Srivastava and Buschmann etc.. These operators play an important role in the theory of integral equations and problems concerning Mathematical Physics. In this paper, we shell study the following multidimensional fractional integral operators having the general function ϕ as their kernel. Also, for the sake of brevity, we shall use the symbol $f(x)$ to represent $f(x_1, x_2, \ldots, x_r)$.

The multivariable I -function introduced by Prasad will be define and represent it in the following manner :

$$I[z_1, \ldots, z_r] = I^{0, n_2 : \ldots : 0, n_r : (m', n') : \ldots : (m^{(r)}, n^{(r)})}_{p_2, q_2 : \ldots : p_r, q_r : (p', q') : \ldots : (p^{(r)}, q^{(r)})}$$

$$\left[z_1, \ldots, z_r \left| \begin{matrix} (a_{2j}, \alpha'_{2j}, \alpha''_{2j})_{1,p_2} : \ldots : (\alpha_{rj}, \alpha'_{rj}, \ldots, \alpha^{(r)}_{rj})_{1,p_r} : (a'_j, \alpha'_j)_{1,p'} : \ldots : (a^{(r)}_j, \alpha^{(r)}_j)_{1,p^{(r)}} \\ (b_{2j}, \beta'_{2j}, \beta''_{2j})_{1,q_2} : \ldots : (b_{rj}, \beta'_{rj}, \ldots, \beta^{(r)}_{rj})_{1,q_r} : (b'_j, \beta'_j)_{1,q'} : \ldots : (b^{(r)}_j, \beta^{(r)}_j)_{1,q^{(r)}} \end{matrix} \right. \right]$$

$$= \frac{1}{(2\pi w)^r} \int_{L_1} \ldots \int_{L_r} \phi_1(s_1) \ldots \phi_r(s_r) \psi(s_1, \ldots, s_r) z_1^{s_1}, \ldots, z_r^{s_r} ds_1 \ldots ds_r \qquad (2.12.1)$$

Where $w = \sqrt{(-1)}$

$$\phi_i(s_i) = \frac{\prod_{j=1}^{m^{(i)}} \Gamma\left(b_j^{(i)} - \beta_j^{(i)} s_i\right) \prod_{j=1}^{n^{(i)}} \Gamma\left(1 - a_j^{(i)} + \alpha_j^{(i)} s_i\right)}{\prod_{j=m^{(i)}+1}^{q^{(i)}} \Gamma\left(1 - b_j^{(i)} + \beta_j^{(i)} s_i\right) \prod_{j=n^{(i)}+1}^{p^{(i)}} \Gamma\left(a_j^{(i)} - \alpha_j^{(i)} s_i\right)} \quad \forall i \in (1, 2, \ldots, r) \qquad (2.12.2)$$

$$\psi(s_1, \ldots, s_r) = \frac{\prod_{j=1}^{n_2} \Gamma\left(1 - a_{2j} + \sum_{i=1}^{2} \alpha_{2j}^{(i)} s_i\right) \prod_{j=1}^{n_3} \Gamma\left(1 - a_{3j} + \sum_{i=1}^{3} \alpha_{3j}^{(i)} s_i\right)}{\prod_{j=n_2+1}^{p_2} \Gamma\left(a_{2j} - \sum_{i=1}^{2} \alpha_{2j}^{(i)} s_i\right) \prod_{j=n_3+1}^{p_3} \Gamma\left(a_{3j} - \sum_{i=1}^{3} \alpha_{3j}^{(i)} s_i\right)}$$

$$\frac{\ldots \prod_{j=1}^{n_r} \Gamma\left(1 - a_{rj} + \sum_{i=1}^{r} \alpha_{rj}^{(i)} s_i\right)}{\ldots \prod_{j=n_r+1}^{p_r} \Gamma\left(a_{rj} - \sum_{i=1}^{r} \alpha_{rj}^{(i)} s_i\right) \prod_{j=1}^{q_2} \Gamma\left(1 - b_{2j} - \sum_{i=1}^{2} \beta_{2j}^{(i)} s_i\right) \ldots \prod_{j=1}^{q_r} \Gamma\left(1 - b_{rj} - \sum_{i=1}^{r} \beta_{rj}^{(i)} s_i\right)} \qquad (2.12.3)$$

$\alpha_j^{(i)}, \beta_j^{(i)}, \alpha_{kj}^{(i)}, \beta_{kj}^{(i)} (i = 1, \ldots, r)(k = 1, \ldots, r)$ are positive numbers, $a_j^{(i)}, b_j^{(i)} (i = 1, \ldots, r), a_{kj}, b_{kj} (k = 2, \ldots, r)$ are complex numbers and here $m^{(i)}, n^{(i)}, p^{(i)}, q^{(i)} (i = 1, \ldots, r), n_k, p_k, q_k (k = 2, \ldots, r)$ are non-negative integers where $0 \le m^{(i)} \le q^{(i)}, 0 \le n^{(i)} \le p^{(i)}, q^k \ge 0, 0 \le n_k \le p_k$. Here (i) denotes the numbers of dashes. The contours L_i in the complex s_i -plane is of the Mellin-Barnes type which runs from $-w\infty$ to $+w\infty$ with indentations, if necessary, to ensure that all the poles of $\Gamma\left(b_j^{(i)} - \beta_j^{(i)} s_i\right)\left(j = 1, \ldots, m^{(i)}\right)$ are separated from those of $\Gamma\left(1 - a_{rj} + \sum_{i=1}^{r} \alpha_{rj}^{(i)} s_i\right)\left(j = 1, \ldots, n_r\right)$.

For further details and asymptotic expansion of the I -function one can refer by Prasad.

In what follows, the multivariable I -function defined by will be represented in the contracted notation:

$I^{0,n_2:\ldots:0,n_r:(m',n'):\ldots:(m^{(r)},n^{(r)})}_{p_2,q_2:\ldots:p_r,q_r:(p',q'):\ldots:(p^{(r)},q^{(r)})}\left[z_1,\ldots,z_r\right]$ Or simply by $I\left[z_1,\ldots z_r\right]$.

We introduce the fractional integration operators by means of the following integral equations:

$$Y\{f(x)\}=Y\{f(x);t_1,\ldots,t_r;\gamma_1,\ldots,\gamma_r\}=\prod_{i=1}^{r}\left(t_i^{\gamma_i-1}\right)$$

$$\int_0^{t_1}\ldots\int_0^{t_r}\prod_{i=1}^{r}\left\{(x_i)^{\gamma_i}\phi\left(\frac{x_1}{t_1},\ldots,\frac{x_r}{t_r}\right)\right\}f(x)dx_1\ldots dx_r \qquad (2.12.4)$$

And

$$N\{f(x)\}=N\{f(x);t_1,\ldots,t_r;\delta_1,\ldots,\delta_r\}=\prod_{i=1}^{r}\left(t_i^{\delta_i}\right)$$

$$\int_{t_1}^{\infty}\ldots\int_{t_r}^{\infty}\prod_{i=1}^{r}\left\{(x_i)^{-\delta_i-1}\phi\left(\frac{t_1}{x_1},\ldots,\frac{t_r}{x_r}\right)\right\}f(x)dx_1\ldots dx_r \qquad (2.12.5)$$

Where the kernel ϕ is such that the above integrals make sense. The above operators exist under the following conditions:

(i) $p_i\le 1,q_i<\infty,\dfrac{1}{p_i}+\dfrac{1}{q_i}=1,i=1,2,\ldots,r$

(ii) $\mathrm{Re}(\gamma_i)>-\dfrac{1}{q_i};\mathrm{Re}(\delta_i)>-\dfrac{1}{p_i};i=1,2,\ldots,r$

(iii) $f(x)\in L_{p_i}((0,\infty),\ldots,(0,\infty));i=1,2,\ldots,r$

(iv) $0\le m^{(i)}\le q^{(i)},0\le n^{(i)}\le p^{(i)},q^k\ge 0,0\le n_k\le p_k$

The following special case of the multidimensional fractional integral operators involving product of Gauss's hypergeometric functions (p.153,eq. (i) and (ii))will be used in Section 3.

$$I\{f(x)\}=\prod_{i=1}^{r}\left\{\frac{t_i^{\gamma_i-1}}{\Gamma(1-a_i)}\right\}\int_0^{t_1}\ldots\int_0^{t_r}\prod_{i=1}^{r}\left\{(x_i)^{\gamma_i}\,{}_2F_1\left(\alpha_i,\beta_i+m;\beta_i;\frac{x_i}{t_i}\right)\right\}f(x)dx_1\ldots dx_r \qquad (2.12.6)$$

And

$$K\{f(x)\}=\prod_{i=1}^{r}\left\{\frac{t_i^{\delta_i}}{\Gamma(1-\alpha_i)}\right\}\int_{t_1}^{\infty}\ldots\int_{t_r}^{\infty}\prod_{i=1}^{r}\left\{(x_i)^{-\delta_i-1}\,{}_2F_1\left(\alpha_i,\beta_i+m;\beta_i;\frac{t_i}{x_i}\right)\right\}f(x)dx_1\ldots dx_r \qquad (2.12.7)$$

The conditions of existence of these operators follow easily from the conditions given in the paper referred to above.

The generalized multidimensional integral transform T , defined below, will also be required during the course of our study:

$$T\left\{ f(x); s_1, \dots, s_r \right\} = \int_0^\infty \dots \int_0^\infty k\left(s_1 x_1, \dots, s_r x_r\right) f(x) dx_1 \dots dx_r \tag{2.12.8}$$

Where $k\left(s_1 x_1, \dots, s_r x_r\right)$ is the kernel of the transform T and the multiple integral occurring in the equation (2.12.8) is assumed to be convergent.

The following multivariable I -function transform will also be used in the sequel:

$$I\left\{ f(x); s_1, \dots, s_r \right\} = \int_0^\infty \dots \int_0^\infty I^{0,n_2:\dots:0,n_r:(m',n'):\dots:\left(m^{(r)},n^{(r)}\right)}_{p_2,q_2:\dots:p_r,q_r:(p',q'):\dots:\left(p^{(r)},q^{(r)}\right)}$$

$$\left[\begin{matrix} s_1 x_1 \\ \vdots \\ s_r x_r \end{matrix} \left| \begin{matrix} (a_{2j}, \alpha'_{2j}, \alpha''_{2j})_{1,p_2} : \dots : (\alpha_{rj}, \alpha'_{rj}, \dots, \alpha^{(r)}_{rj})_{1,p_r} : (a'_j, \alpha'_j)_{1,p'} : \dots : (a^{(r)}_j, \alpha^{(r)}_j)_{1,p^{(r)}} \\ (b_{2j}, \beta'_{2j}, \beta''_{2j})_{1,q_2} : \dots : (b_{rj}, \beta'_{rj}, \dots, \beta^{(r)}_{rj})_{1,q_r} : (b'_j, \beta'_j)_{1,q'} : \dots : (b^{(r)}_j, \beta^{(r)}_j)_{1,q^{(r)}} \end{matrix} \right. \right] f(x) dx_1 \dots dx_r \tag{2.12.9}$$

The transform defined above will be denoted symbolically as follows:

$$I\left\{ f(x); {}^{0,n_2:\dots:0,n_r:(m',n'):\dots:\left(m^{(r)},n^{(r)}\right),\quad (a_{2j}, \alpha'_{2j}, \alpha''_{2j})_{1,p_2} : \dots : (\alpha_{rj}, \alpha'_{rj}, \dots, \alpha^{(r)}_{rj})_{1,p_r} : (a'_j, \alpha'_j)_{1,p'} : \dots : (a^{(r)}_j, \alpha^{(r)}_j)_{1,p^{(r)}}}_{p_2,q_2:\dots:p_r,q_r:(p',q'):\dots:\left(p^{(r)},q^{(r)}\right), (b_{2j}, \beta'_{2j}, \beta''_{2j})_{1,q_2} : \dots : (b_{rj}, \beta'_{rj}, \dots, \beta^{(r)}_{rj})_{1,q_r} : (b'_j, \beta'_j)_{1,q'} : \dots : (b^{(r)}_j, \beta^{(r)}_j)_{1,q^{(r)}}} ; \begin{matrix} s_1 x_1 \\ \vdots \\ s_r x_r \end{matrix} \right\} \tag{2.12.10}$$

Some Basic Properties

Property 1. If $f(x) \in L_{p_i}((0,\infty), \dots, (0,\infty)); 1 \le p_i \le 2$ (or $f(x) \in M_{p_i}((0,\infty), \dots, (0,\infty)); p_i > 2$,where $M_{p_i}((0,\infty), \dots, (0,\infty))$ denotes the class of all functions $f(x) \in L_{p_i}((0,\infty), \dots, (0,\infty)); p_i > 2$, which are the inverse Mellin transforms of functions belonging to $L_{q_i}((-\infty,\infty), \dots, (-\infty,\infty)); \mathrm{Re}(\gamma_i) > \frac{1}{q_i}, \mathrm{Re}(\delta_i) > \frac{1}{p_i}, \frac{1}{p} + \frac{1}{q} = 1 (i = 1, 2, \dots, r)$ and the multidimensional Mellin transform of the function $f(x)$ exists, then

(a) $M\left[Y\left\{ f(x); t_1, \dots, t_r; \gamma_1, \dots, \gamma_r \right\}; s_1, \dots, s_r \right] =$

$$M\left[f(x); s_1, \dots, s_r \right] N\left\{ 1; t_1, \dots, t_r; \gamma_1 - s_1 + 1, \dots, \gamma_r - s_r + 1 \right\} \tag{2.13.1}$$

(b) $M\left[N\left\{ f(x); t_1, \dots, t_r; \delta_1, \dots, \delta_r \right\}; s_1, \dots, s_r \right] =$

$$M\left[f(x); s_1, \dots, s_r \right] Y\left\{ 1; t_1, \dots, t_r; \delta_1 + s_1 + 1, \dots, \delta_r + s_r + 1 \right\} \tag{2.13.2}$$

Where the symbol M occurring in (2.13.1) and (2.13.2) stands for the multidimensional Mellin transform defined in the following way:

$$M\left\{ f(x); s_1, \dots, s_r \right\} = \int_0^\infty \dots \int_0^\infty f(x) \prod_{i=1}^{r} \left(x_i^{s_i - 1} \right) dx_1 \dots dx_r \tag{2.13.3}$$

Provided that the multiple integral involved in (2.13.3) exists.

Proof: To prove (2.13.1), we use (2.13.3)and (2.12.4) to obtain

$$M\left[Y\left\{f(x)\right\}\right]=\int_0^\infty\ldots\int_0^\infty(t_i^{s_i-1})\left\{\int_0^{t_1}\ldots\int_0^{t_r}\prod_{i=1}^{r}\left(x_i^{\gamma_i}\right)f(x)\ \phi\left(\frac{x_1}{t_1},\ldots,\frac{x_r}{t_r}\right)\right\}dt_1\ldots dt_r \qquad (2.13.4)$$

Now interchanging the orders of t_i and $x_i(i=1,2,\ldots,r)$ integrals in (2.13.4), which is easily seen to be permissible under the conditions stated with (2.13.1) and interpreting the results thus obtained with the help of (2.12.5), we arrive at the required result (2.13.1).The result (2.13.2) can be established in a similar manner.

Property 2. If $f(x)\in L_{p_i}((0,\infty),\ldots,(0,\infty));1\le p_i\le 2$, $g(x)\in L_{q_i}((0,\infty),\ldots,(0,\infty));$,

$\operatorname{Re}(\gamma_i)>\max\left(-\frac{1}{p_i},-\frac{1}{q_i}\right),(i=1,2,\ldots,r)$, then

$$\int_0^\infty\ldots\int_0^\infty f(x)Y\left\{g(x)\right\}dx_1\ldots dx_r=\int_0^\infty\ldots\int_0^\infty g(x)N\left\{f(x)\right\}dx_1\ldots dx_r \qquad (2.13.5)$$

Provided that the multiple integrals involved in (2.13.5) are absolutely convergent.

Property 3.

(a) $Y\left\{f(wx);t_1,\ldots,t_r;\gamma_1,\ldots,\gamma_r\right\}=Y\left\{f(x);t_1w_1,\ldots,t_rw_r;\gamma_1,\ldots,\gamma_r\right\}$ (2.13.6)

(b) $N\left\{f(wx);t_1,\ldots,t_r;\delta_1,\ldots,\delta_r\right\}=W\left\{f(x);t_1w_1,\ldots,t_rw_r;\delta_1,\ldots,\delta_r\right\}$ (2.13.7)

Provided that the multiple integrals involved in (2.13.6) and (2.13.7) are absolutely convergent.

Property 4.

(a) $Y\left\{f(1/x);t_1,\ldots,t_r;\gamma_1,\ldots,\gamma_r\right\}=Y\left\{f(x);1/t_1,\ldots,1/t_r;\gamma_1+1,\ldots,\gamma_r+1\right\}$ (2.13.8)

(b) $N\left\{f(1/x);t_1,\ldots,t_r;\delta_1,\ldots,\delta_r\right\}=W\left\{f(x);1/t_1,\ldots,1/t_r;\delta_1-1,\ldots,\delta_r-1\right\}$ (2.13.9)

Provided that the multiple integrals involved in (2.13.8) and (2.13.9) are absolutely convergent.

Property 5.

(a) $Y\left\{\prod_{i=1}^{r}\left(x_i^{\beta_i}\right)f(x);t_1,\ldots,t_r;\gamma_1,\ldots,\gamma_r\right\}=$

$$\prod_{i=1}^{r}\left(t_i^{\beta_i}\right)Y\left\{f(x);t_1,\ldots,t_r;\gamma_1+\beta_1,..,\gamma_r+\beta_r\right\} \qquad (2.13.10)$$

(b) $N\left\{\prod_{i=1}^{r}\left(x_i^{\beta_i}\right)f(x);t_1,\ldots,t_r;\delta_1,\ldots,\delta_r\right\}=$

$$\prod_{i=1}^{r}\left(t_i^{\beta_i}\right)N\left\{f(x);t_1,\ldots,t_r;\delta_1-\beta_1,..,\delta_r-\beta_r\right\} \qquad (2.13.11)$$

Provided that the multiple integrals involved in (2.13.10) and (2.13.11) are absolutely convergent.

The proofs of the properties 2 to 5 are straight forward and follow from the definitions (2.12.4) and (2.12.5) of the multidimensional fractional integral operators involved therein.

Relationship Between Multidimensional Fractional Integral operators and Multidimensional Integral Transforms

In this section, we shall establish two most general theorems exhibiting interconnections between the fractional integral operators Y and N defined by (2.12.4) and (2.12.5) respectively and the integral transform T defined by (2.12.8). Next, we give two interesting corollaries interconnecting the multidimensional fractional integral operators defined by (2.12.6) and (2.12.7) and the multidimensional I -function transform defined by (2.12.9).

Theorem 1. If

$$\tau\left(s_1,..,s_r\right)=T\left\{\psi(u^{\rho})g(u);s_1,...,s_r\right\}=$$

$$\int_0^\infty...\int_0^\infty k(su)\psi(u^{\rho})g(u)du_1...du_r \tag{2.14.1}$$

And

$$\psi\left(t_1,...,t_r\right)=Y\left\{f(x^{\sigma});t_1,...,t_r\right\}=$$

$$\prod_{i=1}^{r}\left(x_i^{-\gamma_i-1}\right)\int_0^{t_1}...\int_0^{t_r} f(x^{\sigma})\prod_{i=1}^{r}\left(x_i^{\gamma_i}\right)\phi\left(\frac{x_1}{t_1},...,\frac{x_r}{t_r}\right)\Bigg\}dx_1...dx_r \tag{2.14.2}$$

Then

$$\tau\left(s_1,..,s_r\right)=\frac{1}{\rho_1...\rho_r}\int_0^\infty...\int_0^\infty f(x^{\sigma})\;\theta(s_1,...,s_r,x_1,...,x_r,\rho_1,...,\rho_r)dx_1...dx_r \qquad (2.143)$$

Where

$$\theta(s_1,...,s_r,x_1,...,x_r,\rho_1,...,\rho_r)=$$

$$Y\left\{\prod_{i=1}^{r}\left(x_i^{\frac{1}{\rho_i}-1}\right)g\left(x^{\frac{1}{\rho}}\right)k\left(sx^{\frac{1}{\rho}}\right);t_1,...,t_r\right\}=$$

$$\prod_{i=1}^{r}\left(t_i^{\gamma_i}\right)\int_{t_1}^{\infty}...\int_{t_r}^{\infty}\prod_{i=1}^{r}\left(x_i^{\frac{1}{\rho_i}-\gamma_i-2}\right)g\left(x^{\frac{1}{\rho}}\right)k\left(sx^{\frac{1}{\sigma}}\right)\;\phi\left(\frac{t_1}{x_1},...,\frac{t_r}{x_r}\right)\Bigg\}dx_1...dx_r \tag{2.14.4}$$

Where ρ_i and $\sigma_i (i = 1, 2, \ldots, r)$ are non zero real numbers of the same sign and all the integrals involved in equations (2.14.1) to (2.14.4) are assumed to be absolutely convergent. Also in (2.14.4) $k\left(sx^{\frac{1}{\sigma}}\right)$ stands for $k\left(s_1 x_1^{\frac{1}{\rho_1}}, \ldots, s_r x_r^{\frac{1}{\rho_r}}\right)$ and so on.

Proof: Applying the formula (2.13.5) to the pair of equations (2.14.2) and (2.14.4), we get

$$\int_0^\infty \ldots \int_0^\infty f(x^\sigma)\theta(s_1, \ldots, s_r, x_1, \ldots, x_r, \rho_1, \ldots, \rho_r) dx_1 \ldots dx_r =$$

$$\int_0^\infty \ldots \int_0^\infty \prod_{i=1}^r \left(x_i^{\frac{1}{\rho_i} - 1} \right) g\left(x^{\frac{1}{\rho}} \right) k\left(sx^{\frac{1}{\sigma}} \right) \psi(x) dx_1 \ldots dx_r \qquad (2.14.5)$$

Now changing the variables of integrations on the right hand side of (2.14.5) slightly and interpreting the result thus obtained with the help of (2.14.1), we easily arrive at (2.14.3) after a little simplification.

If in the above theorem, we replace ρ_i and $h_i (i = 1, 2, \ldots, r)$ and take

$$g(x) = \prod_{i=1}^r \left\{ x_i^{h_i(c_i+1)-1} \right\}, \quad \phi(x) = \prod_{i=1}^r \left\{ \frac{1}{\Gamma(1-\alpha_i)} {}_2F_1(\alpha_i, \beta_i + m; \beta_i; x_i) \right\}$$

And T to be multidimensional I -function transform defined by (2.12.8), the right hand side of equation (2.14.4) assumes the following form:

$$\prod_{i=1}^r \left\{ \frac{t_i^{\gamma_i}}{\Gamma(1-\alpha_i)} \right\} \int_{t_1}^\infty \ldots \int_{t_r}^\infty \prod_{i=1}^r \left\{ x_i^{c_i - \gamma_i - 1} {}_2F_1\left(\alpha_i, \beta_i + m; \beta; \frac{t_i}{x_i} \right) \right\}$$

$$I\left(s_1 x_1^{\frac{1}{h_1}}, \ldots, s_r x_r^{\frac{1}{h_r}} \right) dx_1 \ldots dx_r \qquad (2.14.6)$$

On evaluation the above integral with the help of known results (p.398,eq.(2)) and (p.105,eq.(1)) and the definition of multivariable I -function (2.12.1) and substituting the values of $\theta(s_1, \ldots, s_r, x_1, \ldots, x_r, \rho_1, \ldots, \rho_r)$ thus obtained, in the right hand side of equation (2.14.3), we obtain the following interesting corollary with the help of equations (2.14.1) and (2.14.3).

Corollary 1. If $h_i > 0, \mathrm{Re}(1-\alpha_i) > m, m \in W$,(the set of whole number) $\sigma_i (i = 1, 2, \ldots, r)$ are non-zero real numbers of the same sign.

$$\beta_i \neq 0, -1, -2, \ldots; \mathrm{Re}(C_i) \geq 0$$

$$f(x) = \begin{matrix} o\left(x_i^{A_i}\right), \text{ for small values of } x_i \\ o\left(x_i^{B_i} e^{-C_i x_i}\right) \text{ for large values of } x_i \end{matrix} ; i = 1, 2, \ldots, r$$

The multivariable I -function satisfy the conditions corresponding appropriately to those given by Srivastava and Panda (p.130, eqs. (2.12.6) and (2.12.10)), then

$$I\left\{\prod_{i=1}^{r} x_i^{h_i(c_i+1)-1}\psi(x^h);{}^{0,n_2:\ldots:0,n_r:(m',n'):\ldots:\left(m^{(r)},n^{(r)}\right),}_{p_2,q_2:\ldots:p_r,q_r:(p',q'):\ldots:\left(p^{(r)},q^{(r)}\right),}\right.$$

$$\begin{matrix}(a_{2j},\alpha'_{2j},\alpha''_{2j})_{1,p_2}:\ldots:(\alpha_{rj},\alpha'_{rj},\ldots,\alpha^{(r)}_{rj})_{1,p_r}:(a'_j,\alpha'_j)_{1,p'-1}:\ldots:(a^{(r)}_j,\alpha^{(r)}_j)_{1,p^{(r)}-1}\\(b_{2j},\beta'_{2j},\beta''_{2j})_{1,q_2}:\ldots:(b_{rj},\beta'_{rj},\ldots,\beta^{(r)}_{rj})_{1,q_r}\ :(b'_j,\beta'_j)_{1,q'-1}:\ldots:(b^{(r)}_j,\beta^{(r)}_j)_{1,q^{(r)}-1}\end{matrix}$$

$$\left.\begin{matrix}\left(1-\alpha_1+\gamma_1-c_1,\frac{1}{h_1}\right),\ldots,\left(1-\alpha_r+\gamma_r-c_r,\frac{1}{h_r}\right)\\\left(1-\beta_1+\gamma_1-m-c_1,\frac{1}{h_1}\right),\ldots,\left(1-\beta_r+\gamma_r-m-c_r,\frac{1}{h_r}\right)\end{matrix};s_1,\ldots,s_r\right\} =$$

$$\frac{1}{\prod_{i=1}^{r}(\beta_i)_m} I\left\{\prod_{i=1}^{r} x_i^{h_i(c_i+1)-1} f(x^{h\sigma});{}^{0,n_2:\ldots:0,n_r:(m',n'):\ldots:\left(m^{(r)},n^{(r)}\right),}_{p_2,q_2:\ldots:p_r,q_r:(p',q'):\ldots:\left(p^{(r)},q^{(r)}\right),}\right.$$

$$\begin{matrix}(a_{2j},\alpha'_{2j},\alpha''_{2j})_{1,p_2}:\ldots:(\alpha_{rj},\alpha'_{rj},\ldots,\alpha^{(r)}_{rj})_{1,p_r}:(a'_j,\alpha'_j)_{1,p'-1}:\ldots:(a^{(r)}_j,\alpha^{(r)}_j)_{1,p^{(r)}-1}\\(b_{2j},\beta'_{2j},\beta''_{2j})_{1,q_2}:\ldots:(b_{rj},\beta'_{rj},\ldots,\beta^{(r)}_{rj})_{1,q_r}\ :(b'_j,\beta'_j)_{1,q'-1}:\ldots:(b^{(r)}_j,\beta^{(r)}_j)_{1,q^{(r)}-1}\end{matrix}$$

$$\left.\begin{matrix}\left(\gamma_1-c_1,\frac{1}{h_1}\right),\ldots,\left(\gamma_r-c_r,\frac{1}{h_r}\right)\\\left(1-\beta_1+\gamma_1-c_1,\frac{1}{h_1}\right),\ldots,\left(1-\beta_r+\gamma_r-c_r,\frac{1}{h_r}\right)\end{matrix};s_1,\ldots,s_r\right\} \qquad (2.14.7)$$

Where $\psi(t_1,\ldots,t_r) = I\{f(x);t_1,\ldots,t_r\} = \prod_{i=1}^{r}\left\{\frac{t_i^{\gamma_i}}{\Gamma(1-\alpha_i)}\right\}$

$$\int_0^{t_1}\ldots\int_0^{t_r}\prod_{i=1}^{r}\left\{x_i^{\gamma_i}\,{}_2F_1\left(\alpha_i,\beta_i+m;\beta_i;\frac{x_i}{t_i}\right)\right\} f(x^{\sigma})dx_1\ldots dx_r \qquad (2.14.8)$$

Provided that $\operatorname{Re}\left(c_i-\gamma_i+\frac{1}{h_i}\frac{b_j^{(i)}-1}{B_j^{(i)}}\right)<0;1\le j\le n^{(i)}$

$$\operatorname{Re}\left(\sigma_i B_i+c_i+1+\frac{1}{h_i}\frac{b_j^{(i)}-1}{B_j^{(i)}}\right)<0;1\le j\le n^{(i)}$$

$$\operatorname{Re}\left(\sigma_i A_i+c_i+1+\frac{1}{h_i}\frac{d_k^{(i)}-1}{D_k^{(i)}}\right)<0;1\le k\le m^{(i)},\ \operatorname{Re}\left(\sigma_i A_i+\gamma_i+1\right)>0;i=1,2,\ldots,r$$

Theorem 2. If

$$\tau(s_1,\ldots,s_r)=T\left\{\psi(u^{\rho})g(u);s_1,\ldots,s_r\right\}=$$

$$\int_0^{\infty}\ldots\int_0^{\infty}k(su)\psi(u^{\rho})g(u)du_1\ldots du_r \qquad (2.14.9)$$

And

$$\psi\left(t_1,...,t_r\right)=N\left\{f(x^\sigma);t_1,...,t_r\right\}=$$

$$\prod_{i=1}^{r}\left(t_i^{\gamma_i}\right)\int_{t_1}^{\infty}...\int_{t_r}^{\infty}f(x^\sigma)\prod_{i=1}^{r}\left(x_i^{-\gamma_i-1}\right)\phi\left(\frac{t_1}{x_1},...,\frac{t_r}{x_r}\right)\Bigg\}dx_1...dx_r \qquad (2.14.10)$$

then

$$\tau\left(s_1,..,s_r\right)=\frac{1}{\rho_1...\rho_r}\int_0^\infty...\int_0^\infty f(x^\sigma)\ \ \theta(s_1,...,s_r,t_1,...,t_r,\rho_1,...,\rho_r)dx_1...dx_r \qquad (2.14.11)$$

Where $\theta(s_1,...,s_r,t_1,...,t_r,\rho_1,...,\rho_r)=$

$$Y\left\{\prod_{i=1}^{r}\left(x_i^{\frac{1}{\rho_i}-1}\right)g\left(x^{\frac{1}{\rho}}\right)k\left(sx^{\frac{1}{\rho}}\right);t_1,...,t_r\right\}=$$

$$\prod_{i=1}^{r}\left(t_i^{-\gamma_i-1}\right)\int_0^{t_1}...\int_0^{t_r}\prod_{i=1}^{r}\left(x_i^{\frac{1}{\rho_i}+\gamma_i-1}\right)g\left(x^{\frac{1}{\rho}}\right)k\left(sx^{\frac{1}{\rho}}\right)\ \phi\left(\frac{x_1}{t_1},...,\frac{x_r}{t_r}\right)dx_1...dx_r \qquad (2.14.12)$$

Where ρ_i and $\sigma_i\,(i=1,2,...,r)$ are non zero real numbers of the same sign and all the integrals involved in equations (2.14.1) to (2.14.4) are assumed to be absolutely convergent.

Proof: The proof of the above theorem can be easily developed on the lines similar to those of theorem 1.

Again, if in theorem 2, we replace ρ_i by $h_i\,(i=1,2,...,r)$ and take

$$g(x)=\prod_{i=1}^{r}\left\{x_i^{h_i(c_i+1)-1}\right\},\quad \phi(x)=\prod_{i=1}^{r}\left\{\frac{1}{\Gamma\left(1-\alpha_i\right)}\,{}_2F_1\left(\alpha_i,\beta_i+m;\beta_i;x_i\right)\right\}$$

And T to be multidimensional I -function transform defined by (2.12.8), then proceeding on the lines similar to those of corollary 1, we obtain the following interesting corollary:

Corollary 2. . If $h_i>0,\operatorname{Re}(1-\alpha_i)>m,m\in W$,(the set of whole number) $\sigma_i\,(i=1,2,...,r)$ are non-zero real numbers of the same sign.

$\beta_i\neq 0,-1,-2,...;\operatorname{Re}(c_i)\geq 0$

$$f(x)=\begin{matrix}o\left(x_i^{A_i}\right), \text{for small values of } x_i\\ o\left(x_i^{B_i}e^{-C_ix_i}\right) \text{for large values of } x_i\end{matrix};i=1,2,...,r$$

The multivariable I -function satisfy the conditions corresponding appropriately to those given by Srivastava and Panda (p.130, eqs. (1.6) and (1.10)), then

$$I\left\{\prod_{i=1}^{r}x_i^{h_i(c_i+1)-1}\psi(x^h);{}^{0,n_2:...:0,n_r:(m',n'):...:\left(m^{(r)},n^{(r)}\right),}_{p_2,q_2:...:p_r,q_r:(p',q'):...:\left(p^{(r)},q^{(r)}\right),}\right.$$

$$\begin{matrix}(a_{2j},\alpha'_{2j},\alpha''_{2j})_{1,p_2}:\ldots:(\alpha_{rj},\alpha'_{rj},\ldots,\alpha^{(r)}_{rj})_{1,p_r}:(a'_j,\alpha'_j)_{1,p'-1}:\ldots:(a^{(r)}_j,\alpha^{(r)}_j)_{1,p^{(r)}-1}\\ (b_{2j},\beta'_{2j},\beta''_{2j})_{1,q_2}:\ldots:(b_{rj},\beta'_{rj},\ldots,\beta^{(r)}_{rj})_{1,q_r}\;:(b'_j,\beta'_j)_{1,q'-1}:\ldots:(b^{(r)}_j,\beta^{(r)}_j)_{1,q^{(r)}-1}\end{matrix}$$

$$\left.\begin{matrix}\left(-\gamma_1-c_1,\frac{1}{h_1}\right),\ldots,\left(-\gamma_r-c_r,\frac{1}{h_r}\right)\\ \left(\beta_1-\gamma_1-c_r-1,\frac{1}{h_1}\right),\ldots,\left(\beta_r-\gamma_r-c_r-1,\frac{1}{h_r}\right)\end{matrix};s_1,\ldots,s_r\right\} =$$

$$\frac{1}{\prod_{i=1}^{r}(\beta_i)_m} I\left\{\prod_{i=1}^{r} x_i^{h_i(c_i+1)-1} f(x^{h\sigma});\begin{matrix}0,n_2:\ldots:0,n_r:(m',n'):\ldots:\left(m^{(r)},n^{(r)}\right),\\ p_2,q_2:\ldots:p_r,q_r:(p',q'):\ldots:\left(p^{(r)},q^{(r)}\right),\end{matrix}\right.$$

$$\begin{matrix}(a_{2j},\alpha'_{2j},\alpha''_{2j})_{1,p_2}:\ldots:(\alpha_{rj},\alpha'_{rj},\ldots,\alpha^{(r)}_{rj})_{1,p_r}:(a'_j,\alpha'_j)_{1,p'-1}:\ldots:(a^{(r)}_j,\alpha^{(r)}_j)_{1,p^{(r)}-1}\\ (b_{2j},\beta'_{2j},\beta''_{2j})_{1,q_2}:\ldots:(b_{rj},\beta'_{rj},\ldots,\beta^{(r)}_{rj})_{1,q_r}\;:(b'_j,\beta'_j)_{1,q'-1}:\ldots:(b^{(r)}_j,\beta^{(r)}_j)_{1,q^{(r)}-1}\end{matrix}$$

$$\left.\begin{matrix}\left(\alpha_1-\gamma_1-c_1-1,\frac{1}{h_1}\right),\ldots,\left(\alpha_r-\gamma_r-c_r-1,\frac{1}{h_r}\right)\\ \left(\beta_1-\gamma_1-c_1+m-1,\frac{1}{h_1}\right),\ldots,\left(\beta_r-\gamma_r-c_r+m-1,\frac{1}{h_r}\right)\end{matrix};s_1,\ldots,s_r\right\} \tag{2.14.13}$$

Where $\psi(t_1,\ldots,t_r)=N\{f(x);t_1,\ldots,t_r\}=\prod_{i=1}^{r}\left\{\frac{t_i^{\gamma_i}}{\Gamma(1-\alpha_i)}\right\}$

$$\int_{t_1}^{\infty}\ldots\int_{t_r}^{\infty}\prod_{i=1}^{r}\left\{x_i^{-\gamma_i-1}\,{}_2F_1\left(\alpha_i,\beta_i+m;\beta_i;\frac{t_i}{x_i}\right)\right\} f(x^{\sigma})dx_1\ldots dx_r \tag{2.14.14}$$

Provided that $\operatorname{Re}\left(c_i+\gamma_i+\frac{1}{h_i}\frac{d_j^{(i)}}{D_j^{(i)}}\right)>0;1\le j\le m^{(i)}$

$$\operatorname{Re}\left(\sigma_i B_i+c_i+1+\frac{1}{h_i}\frac{b_j^{(i)}-1}{B_j^{(i)}}\right)<0;1\le j\le n^{(i)};i=1,2,\ldots,r$$

$$\operatorname{Re}\left(\sigma_i A_i+c_i+1+\frac{1}{h_i}\frac{d_k^{(i)}}{D_k^{(i)}}\right)>0;1\le k\le m^{(i)},\ \operatorname{Re}\left(\sigma_i A_i+\beta_i-\gamma_i+m\right)>0;i=1,2,\ldots,r$$

The one and two-dimensional analogous of Theorems 1 and 2 can easily be deduced but since the theorems contain a reasonably detailed analysis of the multidimensional case we prefer to omit their details. The corollaries 1 and 2 given earlier are also new. They give rise to interesting theorems involving multidimensional analogues of fractional integral operators defined by Kober , Riemann-Liouville and Weyl and simpler multidimensional integral transforms, on suitably specializing the fractional integral operators and multidimensional I -function transform involved therein. Again, the one and two-dimensional analogues of corollaries 1 and 2, yield theorems essentially similar to those given earlier by Gupta, Goyal and Handa (p.165-170), Handa (p. 200-204), Kalla (p. 1008-1010, p. 54-56) and Mathur (p.108-121).

In the present paper, we study certain multidimensional fractional integral operators involving a general A -function in their kernel. We give five basic properties of these operators, and then establish two theorems and two corollaries, which are believed to be new. These basic theorems exhibit structural relationships between the multidimensional integral transforms. The one-and –two –dimensional analogues of these results, which are new and of interest in themselves, can easily be deduced. Special cases of these latter theorems will give rise to certain known results obtained from time to time by several earlier authors.

Introduction

Fractional integral operators have been defined and studied by various authors notably by Riemann and Liouville , Weyl, Erdelyi , Kober, Sneddon , Kalla , Saxena , Srivastava and Buschmann etc.. These operators play an important role in the theory of integral equations and problems concerning Mathematical Physics. In this paper, we shell study the following multidimensional fractional integral operators having the general function ϕ as their kernel. Also, for the sake of brevity, we shall use the symbol $f(x)$ to represent $f(x_1, x_2, ..., x_r)$.

Gautam and Goyal defined and represented multivariable A-function as follows:

$$A[z_1,...,z_r] = A^{m,n:m_1,n_1;...;m_r,n_r}_{p,q:p_1,q_1;...;p_r,q_r}$$

$$.\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\left| \begin{matrix} (a_j;A_j^{'},...,A_j^{(r)})_{1,p};(c_j^{'},C_j^{'})_{1,p_1};...;(c_j^{(r)},C_j^{(r)})_{1,p_r} \\ (b_j;B_j^{'},...,B_j^{(r)})_{1,q};(d_j^{'},D_j^{'})_{1,q_1};...;(d_j^{(r)},D_j^{(r)})_{1,q_r} \end{matrix}\right.\right]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}....\int_{L_r}\theta_1(s_1)....\theta_r(s_r)\Phi(s_1,....,s_r)z_1^{s_1}...z_r^{s_r}\ .ds_1...ds_r \qquad (2.15.1)$$

Where $\omega = \sqrt{-1}$;

$$\theta_i(s_i) = \frac{\prod_{j=1}^{m_i}\Gamma(d_j^{(i)} - D_j^{(i)}s_i)\prod_{j=1}^{n_i}\Gamma(1-c_j^{(i)} + C_j^{(i)}s_i)}{\prod_{j=m_i+1}^{q_i}\Gamma(1-d_j^{(i)} + D_j^{(i)}s_i)\prod_{j=n_i+1}^{p_i}\Gamma(c_j^{(i)} - C_j^{(i)}s_i)}$$

$$\forall i \in \{1,\ .\ .\ .\ .\ ,r\} \qquad (2.15.2)$$

$$\Phi(s_1,...,s_r) = \frac{\prod_{j=1}^{n}\Gamma(1-a_j + \sum_{i=1}^{r}A_j^{(i)}s_i)\prod_{j=1}^{m}\Gamma(b_j - \sum_{i=1}^{r}B_j^{(i)}s_i)}{\prod_{j=n+1}^{p}\Gamma(a_j - \sum_{i=1}^{r}A_j^{(i)}s_i)\prod_{j=m+1}^{q}\Gamma(1-b_j + \sum_{i=1}^{r}B_j^{(i)}s_i)} \qquad (2.15.3)$$

Here m , n , p , q , m_i , n_i , p_i , and q_i $(i = 1,\ .\ .\ .\ .\ ,r)$ are non-negative integers and all $a_j^{'}s, b_j^{'}s, d_j^{(i)'}s, c_j^{(i)'}s, A_j^{(i)'}s$ and $B_j^{(i)'}s$ are complex numbers.

The multiple integral defining the A-function of r-variables converges absolutely if

$$|\arg(\Omega_i)z_k| < \frac{\pi}{2}\eta_i, \xi_i^* = 0, \eta_i > 0 \qquad (2.15.4)$$

$$\Omega_i = \prod_{j=1}^{p}\{A_j^{(i)}\}^{A_j^{(i)}}\prod_{j=1}^{q}\{B_j^{(i)}\}^{-B_j^{(i)}}\prod_{j=1}^{q_i}\{D_j^{(i)}\}^{D_j^{(i)}}.\prod_{j=1}^{p_i}\{C_j^{(i)}\}^{-C_j^{(i)}}$$

$$,\ \forall i \in \{1,\ .\ .\ .\ .\ ,r\}; \qquad (2.15.5)$$

$$\xi^*_i = I_m\left[\sum_{j=1}^{p}A_j^{(i)} - \sum_{j=1}^{q}B_j^{(i)} + \sum_{j=1}^{q_i}D_j^{(i)} - \sum_{j=1}^{q_i}C_j^{(i)}\right],\ \forall i \in \{1,\ .\ .\ .\ .\ ,r\} \qquad (2.15.6)$$

$$\eta_i = \mathrm{Re}\left[\sum_{j=1}^{n} A_j^{(i)} - \sum_{j=n+1}^{p} A_j^{(i)} + \sum_{j=1}^{m} B_j^{(i)} - \sum_{j=m+1}^{q} B_j^{(i)} + \sum_{j=1}^{m_i} D_j^{(i)} - \sum_{j=m_i+1}^{q_i} D_j^{(i)} + \sum_{j=1}^{n_i} C_j^{(i)} - \sum_{j=n_i+1}^{p_i} C_j^{(i)}\right]$$

$$\forall i \in \{1, \ldots, \mathrm{r}\}; \qquad (2.15.7)$$

If we take $A_j^{'}s, B_j^{'}s, C_j^{'}s$ and $D_j^{'}s$ as real and positive and $m = 0$, the A-function reduces to multivariable H-function of Srivastava and Panda .

We are using the multivariable A-function in the following concise form throughout the text.

$$A[z_1,\ldots,z_r] = A_{p,q:N_r}^{m,n:M_r}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\,\middle|\,\begin{matrix} P : P_r^{(r)} \\ Q : Q_r^{(r)} \end{matrix}\right]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\ldots\int_{L_r} \theta_1(s_1)\ldots\theta_r(s_r)\Phi(s_1,\ldots,s_r) z_1^{s_1}\ldots z_r^{s_r}\, .ds_1\ldots ds_r \qquad (2.15.8)$$

Where $\omega = \sqrt{-1}$; $M_r = m_1, n_1; \ldots; m_r, n_r$; $N_r = p_1, q_1; \ldots; p_r, q_r$;

$P = (a_j; A_j^{'}, \ldots, A_j^{r})_{1,p}$; $Q = (b_j; B_j^{'}, \ldots, B_j^{r})_{1,q}$; $P_r^{(r)} = (c_j^{'}, C_j^{'})_{1,p_1}; \ldots; (c_j^{(r)}, C_j^{(r)})_{1,p_r}$;

And the definition of the functions $\theta_i(s_i)$ $i = 1, \ldots, r$; $\Phi(s_1,\ldots,s_r)$ and the condition of existence of the multivariable A-function are the same as mentioned by Gautam and Goyal.

We introduce the fractional integration operators by means of the following integral equations:

$$Y\{f(x)\} = Y\{f(x); t_1,\ldots,t_r; \gamma_1,\ldots,\gamma_r\} = \prod_{i=1}^{r}\left(t_i^{\gamma_i - 1}\right)$$

$$\int_0^{t_1}\ldots\int_0^{t_r}\prod_{i=1}^{r}\left\{(x_i)^{\gamma_i}\,\phi\left(\frac{x_1}{t_1},\ldots,\frac{x_r}{t_r}\right)\right\} f(x)dx_1\ldots dx_r \qquad (2.15.9)$$

And

$$N\{f(x)\} = N\{f(x); t_1,\ldots,t_r; \delta_1,\ldots,\delta_r\} = \prod_{i=1}^{r}\left(t_i^{\delta_i}\right)$$

$$\int_{t_1}^{\infty}\ldots\int_{t_r}^{\infty}\prod_{i=1}^{r}\left\{(x_i)^{-\delta_i - 1}\,\phi\left(\frac{t_1}{x_1},\ldots,\frac{t_r}{x_r}\right)\right\} f(x)dx_1\ldots dx_r \qquad (2.15.10)$$

Where the kernel ϕ is such that the above integrals make sense. The above operators exist under the following conditions:

(i) $p_i \le 1, q_i < \infty, \dfrac{1}{p_i} + \dfrac{1}{q_i} = 1, i = 1, 2, \ldots, r$

(ii) $\mathrm{Re}(\gamma_i) > -\dfrac{1}{q_i}; \mathrm{Re}(\delta_i) > -\dfrac{1}{p_i}; i = 1, 2, \ldots, r$

(iii) $f(x) \in L_{p_i}((0,\infty),\ldots,(0,\infty)); i = 1, 2, \ldots, r$

(iv) $0 \le m_i \le q_i, 0 \le n_i \le p_i, q_k \ge 0, 0 \le n_k \le p_k$

The following special case of the multidimensional fractional integral operators involving product of Gauss's hypergeometric functions (p.153,eq. (i) and (ii))will be used in Section 3.

$$I\{f(x)\}=\prod_{i=1}^{r}\left\{\frac{t_i^{\gamma_i-1}}{\Gamma(1-a_i)}\right\}\int_0^{t_1}\ldots\int_0^{t_r}\prod_{i=1}^{r}\left\{(x_i)^{\gamma_i}\,{}_2F_1\left(\alpha_i,\beta_i+m;\beta_i;\frac{x_i}{t_i}\right)\right\}f(x)dx_1\ldots dx_r \qquad (2.15.11)$$

And

$$K\{f(x)\}=\prod_{i=1}^{r}\left\{\frac{t_i^{\delta_i}}{\Gamma(1-\alpha_i)}\right\}\int_{t_1}^{\infty}\ldots\int_{t_r}^{\infty}\prod_{i=1}^{r}\left\{(x_i)^{-\delta_i-1}\,{}_2F_1\left(\alpha_i,\beta_i+m;\beta_i;\frac{t_i}{x_i}\right)\right\}f(x)dx_1\ldots dx_r \qquad (2.15.12)$$

The conditions of existence of these operators follow easily from the conditions given in the paper referred to above.

The generalized multidimensional integral transform T, defined below, will also be required during the course of our study:

$$T\{f(x);s_1,\ldots,s_r\}=\int_0^{\infty}\ldots\int_0^{\infty}k\left(s_1x_1,\ldots,s_rx_r\right)f(x)dx_1\ldots dx_r \qquad (2.15.13)$$

Where $k\left(s_1x_1,\ldots,s_rx_r\right)$ is the kernel of the transform T and the multiple integral occurring in the equation (1.8) is assumed to be convergent.

The following multivariable A-function transform will also be used in the sequel:

$$A\{f(x);s_1,\ldots,s_r\}=\int_0^{\infty}\ldots\int_0^{\infty}A^{m,n:m_1,n_1;\ldots;m_r,n_r}_{p,q:p_1,q_1;\ldots;p_r,q_r}$$

$$\cdot\left[\begin{matrix}s_1x_1\\ \vdots\\ s_rx_r\end{matrix}\left|\begin{matrix}(a_j;A_j',\ldots,A_j^{(r)})_{1,p};(c_j',C_j')_{1,p_1};\ldots;(c_j^{(r)},C_j^{(r)})_{1,p_r}\\ (b_j;B_j',\ldots,B_j^{(r)})_{1,q};(d_j',D_j')_{1,q_1};\ldots;(d_j^{(r)},D_j^{(r)})_{1,q_r}\end{matrix}\right.\right]f(x)dx_1\ldots dx_r \qquad (2.15.14)$$

The transform defined above will be denoted symbolically as follows:

$$A\left\{f(x);^{m,n:m_1,n_1;\ldots;m_r,n_r;(a_j;A_j',\ldots,A_j^{(r)})_{1,p};(c_j',C_j')_{1,p_1};\ldots;(c_j^{(r)},C_j^{(r)})_{1,p_r}}_{p,q:p_1,q_1;\ldots;p_r,q_r;(b_j;B_j',\ldots,B_j^{(r)})_{1,q};(d_j',D_j')_{1,q_1};\ldots;(d_j^{(r)},D_j^{(r)})_{1,q_r}};\begin{matrix}s_1x_1\\ \vdots\\ s_rx_r\end{matrix}\right\} \qquad (2.15.15)$$

Some Basic Properties

Property 1. If $f(x)\in L_{p_i}((0,\infty),\ldots,(0,\infty));1\le p_i\le 2$ (or $f(x)\in M_{p_i}((0,\infty),\ldots,(0,\infty));\,p_i>2$,where $M_{p_i}((0,\infty),\ldots,(0,\infty))$ denotes the class of all functions $f(x)\in L_{p_i}((0,\infty),\ldots,(0,\infty));\,p_i>2$, which are the inverse Mellin transforms of functions belonging to $L_{q_i}((-\infty,\infty),\ldots,(-\infty,\infty));\,\mathrm{Re}(\gamma_i)>\frac{1}{q_i},\mathrm{Re}(\delta_i)>\frac{1}{p_i},\frac{1}{p}+\frac{1}{q}=1(i=1,2,\ldots,r)$ and the multidimensional Mellin transform of the function $f(x)$ exists, then

(a) $M\left[Y\left\{f(x);t_1,...,t_r;\gamma_1,...,\gamma_r\right\};s_1,...,s_r\right]=$

$$M\left[f(x);s_1,...,s_r\right]N\left\{1;t_1,...,t_r;\gamma_1-s_1+1,...,\gamma_r-s_r+1\right\} \quad (2.16.1)$$

(b) $M\left[N\left\{f(x);t_1,...,t_r;\delta_1,...,\delta_r\right\};s_1,...,s_r\right]=$

$$M\left[f(x);s_1,...,s_r\right]Y\left\{1;t_1,...,t_r;\delta_1+s_1+1,...,\delta_r+s_r+1\right\} \quad (2.162)$$

Where the symbol M occurring in (2.16.1) and (2.16.2) stands for the multidimensional Mellin transform defined in the following way:

$$M\left\{f(x);s_1,...,s_r\right\}=\int_0^\infty...\int_0^\infty f(x)\prod_{i=1}^{r}\left(x_i^{s_i-1}\right)dx_1...dx_r \quad (2.16.3)$$

Provided that the multiple integral involved in (2.16.3) exists.

Proof: To prove (2.16.1), we use (2.16.3)and (2.15.9) to obtain

$$M\left[Y\left\{f(x)\right\}\right]=\int_0^\infty...\int_0^\infty(t_i^{s_i-1})\left\{\int_0^{t_1}...\int_0^{t_r}\prod_{i=1}^{r}\left(x_i^{\gamma_i}\right)f(x)\ \phi\left(\frac{x_1}{t_1},...,\frac{x_r}{t_r}\right)\right\}dt_1...dt_r \quad (2.16.4)$$

Now interchanging the orders of t_i and $x_i(i=1,2,...,r)$ integrals in (2.16.4), which is easily seen to be permissible under the conditions stated with (2.16.1) and interpreting the results thus obtained with the help of (2.15.10), we arrive at the required result (2.16.1).

The result (2.16.2) can be established in a similar manner.

Property 2. If $f(x)\in L_{p_i}((0,\infty),...,(0,\infty));1\le p_i\le 2$, $g(x)\in L_{q_i}((0,\infty),...,(0,\infty));$,

$\mathrm{Re}(\gamma_i)>\max\left(-\frac{1}{p_i},-\frac{1}{q_i}\right),(i=1,2,...,r)$, then

$$\int_0^\infty...\int_0^\infty f(x)Y\left\{g(x)\right\}dx_1...dx_r=\int_0^\infty...\int_0^\infty g(x)N\left\{f(x)\right\}dx_1...dx_r \quad (2.16.5)$$

Provided that the multiple integrals involved in (2.16.5) are absolutely convergent.

Property 3.

(a) $Y\left\{f(wx);t_1,...,t_r;\gamma_1,...,\gamma_r\right\}=Y\left\{f(x);t_1w_1,...,t_rw_r;\gamma_1,...,\gamma_r\right\}$ (2.16.6)

(b) $N\left\{f(wx);t_1,...,t_r;\delta_1,...,\delta_r\right\}=W\left\{f(x);t_1w_1,...,t_rw_r;\delta_1,...,\delta_r\right\}$ (2.167)

Provided that the multiple integrals involved in (2.16.6) and (2.16.7) are absolutely convergent.

Property 4.

(a) $Y\left\{f(1/x);t_1,...,t_r;\gamma_1,...,\gamma_r\right\}=Y\left\{f(x);1/t_1,...,1/t_r;\gamma_1+1,...,\gamma_r+1\right\}$ (2.16.8)

(b) $N\left\{f(1/x);t_1,...,t_r;\delta_1,...,\delta_r\right\}=W\left\{f(x);1/t_1,...,1/t_r;\delta_1-1,...,\delta_r-1\right\}$ (2.16.9)

Provided that the multiple integrals involved in (2.16.8) and (2.16.9) are absolutely convergent.

Property 5.

(a) $Y\left\{\prod_{i=1}^{r}\left(x_i^{\beta_i}\right)f(x);t_1,...,t_r;\gamma_1,...,\gamma_r\right\}=$

$$\prod_{i=1}^{r}\left(t_i^{\beta_i}\right)Y\left\{f(x);t_1,...,t_r;\gamma_1+\beta_1,..,\gamma_r+\beta_r\right\} \tag{2.16.10}$$

(b) $N\left\{\prod_{i=1}^{r}\left(x_i^{\beta_i}\right)f(x);t_1,...,t_r;\delta_1,...,\delta_r\right\}=$

$$\prod_{i=1}^{r}\left(t_i^{\beta_i}\right)N\left\{f(x);t_1,...,t_r;\delta_1-\beta_1,..,\delta_r-\beta_r\right\} \tag{2.16.11}$$

Provided that the multiple integrals involved in (2.16.10) and (2.16.11) are absolutely convergent.

The proofs of the properties 2 to 5 are straight forward and follow from the definitions (2.15.9) and (2.15.10) of the multidimensional fractional integral operators involved therein.

Relationship Between Multidimensional Fractional Integral operators and Multidimensional Integral Transforms

In this section, we shall establish two most general theorems exhibiting interconnections between the fractional integral operators Y and N defined by (2.15.9) and (2.15.10) respectively and the integral transform T defined by (2.15.13). Next, we give two interesting corollaries interconnecting the multidimensional fractional integral operators defined by (2.15.1) and (2.15.12) and the multidimensional A-function transform defined by (2.15.14).

Theorem 1. If

$$\tau\left(s_1,..,s_r\right)=T\left\{\psi(u^{\rho})g(u);s_1,...,s_r\right\}=\int_0^{\infty}...\int_0^{\infty}k(su)\psi(u^{\rho})g(u)du_1...du_r \tag{2.17.1}$$

And $\psi\left(t_1,...,t_r\right)=Y\left\{f(x^{\sigma});t_1,...,t_r\right\}=$

$$\prod_{i=1}^{r}\left(x_i^{-\gamma_i-1}\right)\int_0^{t_1}...\int_0^{t_r}f(x^{\sigma})\prod_{i=1}^{r}\left(x_i^{\gamma_i}\right)\phi\left(\frac{x_1}{t_1},...,\frac{x_r}{t_r}\right)\Bigg\}dx_1...dx_r \tag{2.17.2}$$

Then

$$\tau\left(s_1,..,s_r\right)=\frac{1}{\rho_1...\rho_r}\int_0^{\infty}...\int_0^{\infty}f(x^{\sigma})\ \theta(s_1,...,s_r,x_1,...,x_r,\rho_1,...,\rho_r)dx_1...dx_r \tag{2.17.3}$$

Where $\theta(s_1,...,s_r,x_1,...,x_r,\rho_1,...,\rho_r)=$

$$Y\left\{\prod_{i=1}^{r}\left(x_i^{\frac{1}{\rho_i}-1}\right)g\left(x^{\frac{1}{\rho}}\right)k\left(sx^{\frac{1}{\rho}}\right);t_1,...,t_r\right\}=$$

$$\prod_{i=1}^{r}\left(t_i^{\gamma_i}\right)\int_{t_1}^{\infty}...\int_{t_r}^{\infty}\prod_{i=1}^{r}\left(x_i^{\frac{1}{\rho_i}-\gamma_i-2}\right)g\left(x^{\frac{1}{\rho}}\right)k\left(sx^{\frac{1}{\sigma}}\right)\ \phi\left(\frac{t_1}{x_1},...,\frac{t_r}{x_r}\right)\Bigg\}dx_1...dx_r \qquad (2.17.4)$$

Where ρ_i and $\sigma_i\,(i=1,2,...,r)$ are non zero real numbers of the same sign and all the integrals involved in equations (2.17.1) to (2.17.4) are assumed to be absolutely convergent. Also in (2.17.4) $k\left(sx^{\frac{1}{\sigma}}\right)$ stands for $k\left(s_1x_1^{\frac{1}{\rho_1}},...,s_rx_r^{\frac{1}{\rho_r}}\right)$ and so on.

Proof: Applying the formula (2.16.5) to the pair of equations (2.17.2) and (2.17.4), we get

$$\int_0^{\infty}...\int_0^{\infty}f(x^{\sigma})\theta(s_1,...,s_r,x_1,...,x_r,\rho_1,...,\rho_r)dx_1...dx_r=$$

$$\int_0^{\infty}...\int_0^{\infty}\prod_{i=1}^{r}\left(x_i^{\frac{1}{\rho_i}-1}\right)g\left(x^{\frac{1}{\rho}}\right)k\left(sx^{\frac{1}{\sigma}}\right)\psi(x)dx_1...dx_r \qquad (2.17.5)$$

Now changing the variables of integrations on the right hand side of (2.17.5) slightly and interpreting the result thus obtained with the help of (2.17.1), we easily arrive at (2.17.3) after a little simplification.

If in the above theorem, we replace ρ_i and $h_i\,(i=1,2,...,r)$ and take

$$g(x)=\prod_{i=1}^{r}\left\{x_i^{h_i(c_i+1)-1}\right\},\qquad \phi(x)=\prod_{i=1}^{r}\left\{\frac{1}{\Gamma(1-\alpha_i)}\,{}_2F_1\left(\alpha_i,\beta_i+m;\beta_i;x_i\right)\right\}$$

And T to be multidimensional A-function transform defined by (2.15.13), the right hand side of equation (2.17.4) assumes the following form:

$$\prod_{i=1}^{r}\left\{\frac{t_i^{\gamma_i}}{\Gamma(1-\alpha_i)}\right\}\int_{t_1}^{\infty}...\int_{t_r}^{\infty}\prod_{i=1}^{r}\left\{x_i^{c_i-\gamma_i-1}\,{}_2F_1\left(\alpha_i,\beta_i+m;\beta;\frac{t_i}{x_i}\right)\right\}$$

$$I\left(s_1x_1^{\frac{1}{h_1}},...,s_rx_r^{\frac{1}{h_r}}\right)dx_1...dx_r \qquad (2.17.6)$$

On evaluation the above integral with the help of known results (p.398,eq.(2)) and (p.105,eq.(1)) and the definition of multivariable A-function (2.15.1) and substituting the values of $\theta(s_1,...,s_r,x_1,...,x_r,\rho_1,...,\rho_r)$ thus obtained, in the right hand side of equation (2.17.3), we obtain the following interesting corollary with the help of equations (2.17.1) and (2.17.3).

Corollary 1. If $h_i > 0, \mathrm{Re}(1-\alpha_i) > m, m \in W$,(the set of whole number) $\sigma_i (i = 1, 2, ..., r)$ are non-zero real numbers of the same sign. $\beta_i \neq 0, -1, -2, ...; \mathrm{Re}(C_i) \geq 0$

$$f(x) = \frac{o\left(x_i^{A_i}\right), \text{ for small values of } x_i}{o\left(x_i^{B_i} e^{-C_i x_i}\right) \text{ for large values of } x_i}; i = 1, 2, ..., r$$

The multivariable A-function satisfy the conditions corresponding appropriately to those given by Srivastava and Panda (p.130, eqs. (2.15.1) and (2.15.15)), then

$$I\left\{\prod_{i=1}^{r} x_i^{h_i(c_i+1)-1} \psi(x^h);^{0,n_2:...:0,n_r:(m',n'):...:\left(m^{(r)},n^{(r)}\right),}_{p_2,q_2:...:p_r,q_r:(p',q'):...:\left(p^{(r)},q^{(r)}\right),}\right.$$

$$\begin{matrix}(a_{2j},\alpha'_{2j},\alpha''_{2j})_{1,p_2}:...:(\alpha_{rj},\alpha'_{rj},...,\alpha^{(r)}_{rj})_{1,p_r}:(a'_j,\alpha'_j)_{1,p'-1}:...:(a^{(r)}_j,\alpha^{(r)}_j)_{1,p^{(r)}-1} \\ (b_{2j},\beta'_{2j},\beta''_{2j})_{1,q_2}:...:(b_{rj},\beta'_{rj},...,\beta^{(r)}_{rj})_{1,q_r}:(b'_j,\beta'_j)_{1,q'-1}:...:(b^{(r)}_j,\beta^{(r)}_j)_{1,q^{(r)}-1}\end{matrix}$$

$$\left.\begin{matrix}\left(1-\alpha_1+\gamma_1-c_1,\frac{1}{h_1}\right),...,\left(1-\alpha_r+\gamma_r-c_r,\frac{1}{h_r}\right) \\ \left(1-\beta_1+\gamma_1-m-c_1,\frac{1}{h_1}\right),...,\left(1-\beta_r+\gamma_r-m-c_r,\frac{1}{h_r}\right)\end{matrix}; s_1,...,s_r\right\} =$$

$$\frac{1}{\prod_{i=1}^{r}(\beta_i)_m} I\left\{\prod_{i=1}^{r} x_i^{h_i(c_i+1)-1} f(x^{h\sigma});^{0,n_2:...:0,n_r:(m',n'):...:\left(m^{(r)},n^{(r)}\right),}_{p_2,q_2:...:p_r,q_r:(p',q'):...:\left(p^{(r)},q^{(r)}\right),}\right.$$

$$\begin{matrix}(a_{2j},\alpha'_{2j},\alpha''_{2j})_{1,p_2}:...:(\alpha_{rj},\alpha'_{rj},...,\alpha^{(r)}_{rj})_{1,p_r}:(a'_j,\alpha'_j)_{1,p'-1}:...:(a^{(r)}_j,\alpha^{(r)}_j)_{1,p^{(r)}-1} \\ (b_{2j},\beta'_{2j},\beta''_{2j})_{1,q_2}:...:(b_{rj},\beta'_{rj},...,\beta^{(r)}_{rj})_{1,q_r}:(b'_j,\beta'_j)_{1,q'-1}:...:(b^{(r)}_j,\beta^{(r)}_j)_{1,q^{(r)}-1}\end{matrix}$$

$$\left.\begin{matrix}\left(\gamma_1-c_1,\frac{1}{h_1}\right),...,\left(\gamma_r-c_r,\frac{1}{h_r}\right) \\ \left(1-\beta_1+\gamma_1-c_1,\frac{1}{h_1}\right),...,\left(1-\beta_r+\gamma_r-c_r,\frac{1}{h_r}\right)\end{matrix}; s_1,...,s_r\right\} \qquad (2.17.7)$$

Where $\psi(t_1,...,t_r) = I\{f(x); t_1,...,t_r\} = \prod_{i=1}^{r}\left\{\frac{t_i^{\gamma_i}}{\Gamma(1-\alpha_i)}\right\}$

$$\int_0^{t_1}...\int_0^{t_r}\prod_{i=1}^{r}\left\{x_i^{\gamma_i}\,{}_2F_1\left(\alpha_i,\beta_i+m;\beta_i;\frac{x_i}{t_i}\right)\right\} f(x^\sigma)dx_1...dx_r \qquad (2.17.8)$$

Provided that

$$\mathrm{Re}\left(c_i-\gamma_i+\frac{1}{h_i}\frac{b_j^{(i)}-1}{B_j^{(i)}}\right)<0; 1\leq j\leq n_i,\ \mathrm{Re}\left(\sigma_i B_i+c_i+1+\frac{1}{h_i}\frac{b_j^{(i)}-1}{B_j^{(i)}}\right)<0; 1\leq j\leq n_i$$

$$\mathrm{Re}\left(\sigma_i A_i+c_i+1+\frac{1}{h_i}\frac{d_k^{(i)}-1}{D_k^{(i)}}\right)<0; 1\leq k\leq m_i,\ \mathrm{Re}\left(\sigma_i A_i+\gamma_i+1\right)>0; i=1,2,...,r$$

Theorem 2. If

$$\tau(s_1,..,s_r) = T\{\psi(u^\rho)g(u); s_1,...,s_r\} =$$

$$\int_0^\infty \ldots \int_0^\infty k(su)\psi(u^\rho)g(u)du_1 \ldots du_r \tag{2.17.9}$$

And $\psi(t_1,...,t_r) = N\{f(x^\sigma); t_1,...,t_r\} =$

$$\prod_{i=1}^{r}(t_i^{\gamma_i}) \int_{t_1}^\infty \ldots \int_{t_r}^\infty f(x^\sigma) \prod_{i=1}^{r}(x_i^{-\gamma_i-1}) \phi\left(\frac{t_1}{x_1},...,\frac{t_r}{x_r}\right)\Bigg\} dx_1 \ldots dx_r \tag{2.17.10}$$

then

$$\tau(s_1,..,s_r) = \frac{1}{\rho_1 \ldots \rho_r} \int_0^\infty \ldots \int_0^\infty f(x^\sigma)\ \theta(s_1,...,s_r,t_1,...,t_r,\rho_1,...,\rho_r) dx_1 \ldots dx_r \tag{2.17.11}$$

Where $\theta(s_1,...,s_r,t_1,...,t_r,\rho_1,...,\rho_r) =$

$$Y\left\{\prod_{i=1}^{r}\left(x_i^{\frac{1}{\rho_i}-1}\right) g\left(x^{\frac{1}{\rho}}\right) k\left(sx^{\frac{1}{\rho}}\right); t_1,...,t_r\right\} =$$

$$\prod_{i=1}^{r}\left(t_i^{-\gamma_i-1}\right)\int_0^{t_1}\ldots\int_0^{t_r}\prod_{i=1}^{r}\left(x_i^{\frac{1}{\rho_i}+\gamma_i-1}\right) g\left(x^{\frac{1}{\rho}}\right) k\left(sx^{\frac{1}{\rho}}\right)\ \phi\left(\frac{x_1}{t_1},...,\frac{x_r}{t_r}\right) dx_1 \ldots dx_r \tag{2.17.12}$$

Where ρ_i and $\sigma_i (i = 1, 2,..., r)$ are non zero real numbers of the same sign and all the integrals involved in equations (2.17.1) to (2.17.4) are assumed to be absolutely convergent.

Proof: The proof of the above theorem can be easily developed on the lines similar to those of theorem 1.

Again, if in theorem 2, we replace ρ_i by $h_i (i = 1, 2,..., r)$ and take

$$g(x) = \prod_{i=1}^{r}\left\{x_i^{h_i(c_i+1)-1}\right\} \quad , \quad \phi(x) = \prod_{i=1}^{r}\left\{\frac{1}{\Gamma(1-\alpha_i)}\, {}_2F_1(\alpha_i, \beta_i + m; \beta_i; x_i)\right\}$$

And T to be multidimensional A-function transform defined by (2.15.13), then proceeding on the lines similar to those of corollary 1, we obtain the following interesting corollary:

Corollary 2. . If $h_i > 0, \mathrm{Re}(1-\alpha_i) > m, m \in W$,(the set of whole number) $\sigma_i (i = 1, 2,..., r)$ are non-zero real numbers of the same sign.

$$\beta_i \neq 0, -1, -2, ...; \mathrm{Re}(c_i) \geq 0\ ;\ f(x) = \begin{matrix} o\left(x_i^{A_i}\right), \text{ for small values of } x_i \\ o\left(x_i^{B_i} e^{-C_i x_i}\right) \text{ for large values of } x_i \end{matrix} ; i = 1, 2, ..., r$$

The multivariable A-function satisfy the conditions corresponding appropriately to those given by Srivastava and Panda (p.130, eqs. (1.1) and (1.15)), then

$$I\left\{\prod_{i=1}^{r} x_i^{h_i(c_i+1)-1}\psi(x^h);\ {}^{0,n_2:...:0,n_r:(m',n'):...:(m^{(r)},n^{(r)}),}_{p_2,q_2:...:p_r,q_r:(p',q'):...:(p^{(r)},q^{(r)}),}\right.$$

$$
\begin{matrix} (a_{2j},\alpha'_{2j},\alpha''_{2j})_{1,p_2}:\ldots:(\alpha_{rj},\alpha'_{rj},\ldots,\alpha^{(r)}_{rj})_{1,p_r}:(a'_j,\alpha'_j)_{1,p'-1}:\ldots:(a^{(r)}_j,\alpha^{(r)}_j)_{1,p^{(r)}-1} \\ (b_{2j},\beta'_{2j},\beta''_{2j})_{1,q_2}:\ldots:(b_{rj},\beta'_{rj},\ldots,\beta^{(r)}_{rj})_{1,q_r}\ :(b'_j,\beta'_j)_{1,q'-1}:\ldots:(b^{(r)}_j,\beta^{(r)}_j)_{1,q^{(r)}-1} \end{matrix}
$$

$$
\left.\begin{matrix} \left(-\gamma_1-c_1,\frac{1}{h_1}\right),\ldots,\left(-\gamma_r-c_r,\frac{1}{h_r}\right) \\ \left(\beta_1-\gamma_1-c_r-1,\frac{1}{h_1}\right),\ldots,\left(\beta_r-\gamma_r-c_r-1,\frac{1}{h_r}\right) \end{matrix};s_1,\ldots,s_r\right\} =
$$

$$
\frac{1}{\prod_{i=1}^{r}(\beta_i)_m} I\left\{\prod_{i=1}^{r} x_i^{h_i(c_i+1)-1} f(x^{h\sigma});^{0,n_2:\ldots:0,n_r:(m',n'):\ldots:\left(m^{(r)},n^{(r)}\right),}_{p_2,q_2:\ldots:p_r,q_r:(p',q'):\ldots:\left(p^{(r)},q^{(r)}\right),}\right.
$$

$$
\begin{matrix} (a_{2j},\alpha'_{2j},\alpha''_{2j})_{1,p_2}:\ldots:(\alpha_{rj},\alpha'_{rj},\ldots,\alpha^{(r)}_{rj})_{1,p_r}:(a'_j,\alpha'_j)_{1,p'-1}:\ldots:(a^{(r)}_j,\alpha^{(r)}_j)_{1,p^{(r)}-1} \\ (b_{2j},\beta'_{2j},\beta''_{2j})_{1,q_2}:\ldots:(b_{rj},\beta'_{rj},\ldots,\beta^{(r)}_{rj})_{1,q_r}\ :(b'_j,\beta'_j)_{1,q'-1}:\ldots:(b^{(r)}_j,\beta^{(r)}_j)_{1,q^{(r)}-1} \end{matrix}
$$

$$
\left.\begin{matrix} \left(\alpha_1-\gamma_1-c_1-1,\frac{1}{h_1}\right),\ldots,\left(\alpha_r-\gamma_r-c_r-1,\frac{1}{h_r}\right) \\ \left(\beta_1-\gamma_1-c_1+m-1,\frac{1}{h_1}\right),\ldots,\left(\beta_r-\gamma_r-c_r+m-1,\frac{1}{h_r}\right) \end{matrix};s_1,\ldots,s_r\right\} \qquad (2.17.13)
$$

Where $\psi(t_1,\ldots,t_r)=N\{f(x);t_1,\ldots,t_r\}=\prod_{i=1}^{r}\left\{\frac{t_i^{\gamma_i}}{\Gamma(1-\alpha_i)}\right\}$

$$
\int_{t_1}^{\infty}\ldots\int_{t_r}^{\infty}\prod_{i=1}^{r}\left\{x_i^{-\gamma_i-1}\,{}_2F_1\left(\alpha_i,\beta_i+m;\beta_i;\frac{t_i}{x_i}\right)\right\} f(x^{\sigma})dx_1\ldots dx_r \qquad (2.17.14)
$$

Provided that $\operatorname{Re}\left(c_i+\gamma_i+\frac{1}{h_i}\frac{d_j^{(i)}}{D_j^{(i)}}\right)>0;1\le j\le m_i$

$$
\operatorname{Re}\left(\sigma_i B_i+c_i+1+\frac{1}{h_i}\frac{b_j^{(i)}-1}{B_j^{(i)}}\right)<0;1\le j\le n_i;i=1,2,\ldots,r
$$

$$
\operatorname{Re}\left(\sigma_i A_i+c_i+1+\frac{1}{h_i}\frac{d_k^{(i)}}{D_k^{(i)}}\right)>0;1\le k\le m_i,\ \operatorname{Re}\left(\sigma_i A_i+\beta_i-\gamma_i+m\right)>0;i=1,2,\ldots,r
$$

The one and two-dimensional analogous of Theorems 1 and 2 can easily be deduced but since the theorems contain a reasonably detailed analysis of the multidimensional case we prefer to omit their details. The corollaries 1 and 2 given earlier are also new. They give rise to interesting theorems involving multidimensional analogues of fractional integral operators defined by Kober [8], Riemann-Liouville [4] and Weyl [4] and simpler multidimensional integral transforms, on suitably specializing the fractional integral operators and multidimensional A-function transform involved therein. Again, the one and two-dimensional analogues of corollaries 1 and 2, yield theorems essentially similar to those given earlier by Gupta, Goyal and Handa (p.165-170), Handa (p. 200-204), Kalla (p. 1008-1010, p. 54-56) and Mathur (p.108-121).

CHAPTER 3

ON THE FRACTIONAL INTEGRAL OPERATORS ASSOCIATED WITH CERTAIN GENERALIZED HYPERGEOMETRIC FUNCTION FOR REAL POSITIVE DEFINITE SYMMETRIC MATRIX

In the present chapter, fractional integral operators associated with I -function for real positive symmetric definite matrix have been discussed. These operators have a wide range of applications in the field of Mathematical Physics and Linear differential equations. A number of special cases of our operators have been mentioned.

Introduction

Fractional integration is an immediate generalization of repeated integration. Fractional integral operators occur in the solutions of linear differential equations, partial differential equations and in the integral representations of hypergeometric functions of one or more variables. Riesz (1949) and Garding (1947) respectively introduced Riemann-Liouville integral of vector and matrix variables and applied them in the solution of differential equation associated with Cauchy's problem.

(1.1) I -function with matrix argument

Let X is a $p \times p$ real symmetric positive definite matrix of functionally independent variables. Let the I -function of X be denoted by

The I -function introduced by Saxena (1982) will be represented and defined as follows:

$$I[Z] = I^{m,n}_{p_i,q_i;r}[Z] = I^{m,n}_{p_i,q_i;r}\left[z \left| \begin{matrix} (a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i} \\ (b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i} \end{matrix} \right. \right] = \frac{1}{2\pi\omega}\int_L \theta(s)ds \tag{3.1.1}$$

where $\omega = \sqrt{-1}$

$$\theta(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j - \beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\sum_{i=1}^{r}\left\{ \prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+\beta_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}s)\right\}} \tag{3.1.2}$$

$p_{i,} q_i (i=1,\ldots,r), m, n$ are integers satisfying $0 \le n \le p_i$, $0 \le m \le q_i, (i=1,\ldots,r), r$ is

finite $\alpha_j, \beta_j, \alpha_{ij}, \beta_{ji}$ are real and $a_{j,} b_j, a_{ji}, b_{ji}$ are complex numbers such that

$\alpha_j(b_h+v) \ne \beta_h(a_j - v - k)$ for $v, k = 01, 2, \ldots$

We shall use the following notations:

$A^* = (a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}; B^* = (b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}$

If we take $r=1$ in (3.1.1), the I - function reduces to the Fox's H-function (1965).

It is assumed that $I(XY)=I(YX)$ for real symmetric $m\times m$ positive definite matrices

X and Y, $I(X)$ is defined by the following integral equation:

$$\int\limits_{X>0} |X|^{\rho-\frac{m+1}{2}} I(X)dX =$$

$$\frac{\prod_{j=1}^{m}\Gamma_m(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma_m(\frac{m+1}{2}-a_j+\alpha_j s)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma_m(\frac{m+1}{2}-b_{ji}+\beta_{ji}s)\prod_{j=n+1}^{p_i}\Gamma_m(a_{ji}-\alpha_{ji}s)\right\}} \qquad (3.1.3)$$

Sethi (2003) discussed the following fractional integral operators involving H -function of matrix arguments:

$$R[f(X)]=R\left[{}^{(a_j,\alpha_j)_{1,r},(b_j,\beta_j)_{1,s}}_{\sigma,\rho,\gamma};f(X)\right]=\frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)}$$

$$\int\limits_{0<U<X}|U|^{\sigma}|X-U|^{\rho-\frac{m+1}{2}}H_{r,s}^{p,q}\left[\gamma(I-UX^{-1})\middle|{}^{(a_j,\alpha_j)_{1,r}}_{(b_j,\beta_j)_{1,s}}\right]f(U)dU \qquad (3.1.4)$$

$$K[f(X)]=K\left[{}^{(a_j,\alpha_j)_{1,r},(b_j,\beta_j)_{1,s}}_{\delta,\rho,\gamma};f(X)\right]=\frac{|X|^{\delta}}{\Gamma_m(\rho)}$$

$$\int\limits_{U>X}|U|^{-\delta-\rho}|U-X|^{\rho-\frac{m+1}{2}}H_{r,s}^{p,q}\left[\gamma(I-XU^{-1})\middle|{}^{(a_j,\alpha_j)_{1,r}}_{(b_j,\beta_j)_{1,s}}\right]f(U)dU \qquad (3.1.5)$$

Where $f(X)=f(x_{11},\ldots,x_{1m},x_{21},\ldots,x_{mm})$ be a real bounded function of a complex parameter.

Matrix transform

A generalized matrix transform or M-transform of a function $f(X)$ of a $m\times m$ real symmetric positive definite or strictly negative definite matrix X is defined as follows:

$$M_f(s)=\int\limits_{X>0}|X|^{s-\frac{m+1}{2}}f(X)dX \qquad (X>0) \qquad (3.1.6)$$

Whenever $M_f(s)$ exists. Also $f(X)$ is assumed to be a symmetric function i.e.

$f(BX)=f(XB)=f\left(B^{1/2}XB^{1/2}\right)$ for $B=B'>0$. When $X<0$ replace X by $-X$ in

M -transform.

Integral operators involving I -function

$$Y[f(X)] = Y\left[f(X)\middle|\sigma,\rho,\gamma;_{B^*}^{A^*}\right] = \frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)}$$

$$\int_{0<U<X} |U|^{\sigma} |X-U|^{\rho-\frac{m+1}{2}} I_{P_i,Q_i:r}^{M,N}\left[\gamma(1-UX^{-1})\Big|_{B^*}^{A^*}\right] f(U)dU \qquad (3.1.7)$$

$$N[f(X)] = N\left[f(X)\middle|\delta,\rho,\gamma;_{B^*}^{A^*}\right] = \frac{|X|^{\delta}}{\Gamma_m(\rho)}$$

$$\int_{U>X} |U|^{-\delta-\rho} |U-X|^{\rho-\frac{m+1}{2}} I_{P_i,Q_i:r}^{M,N}\left[\gamma(I-XU^{-1})\Big|_{B^*}^{A^*}\right] f(U)dU \qquad (3.1.8)$$

The above defined operators exists under the following conditions:

(i) $P_i \geq 1, Q_i < \infty, \frac{1}{P_i} + \frac{1}{Q_i} = 1, |\arg(I-a)| < \pi$

(ii) ($\mathrm{Re}(\sigma) > \frac{1}{Q_i}, \mathrm{Re}(\delta) > \frac{1}{P_i}, \mathrm{Re}(\rho) > \frac{m+1}{2}$

(iii) $\mathrm{Re}(\alpha + \min\limits_{1\leq j\leq M} \frac{b_j}{\beta_j}) > \frac{m+1}{2}$, (iv) $f(X) \in L_{P_i}(0,\infty)$.

The last condition ensures that $Y[f(X)]$ and $N[f(X)]$ both exist and also both belong to $L_{P_i}(0,\infty)$.

Main Results

The following theorems of the operators defined by (3.1.7) and (3.1.8) have been established in the expression of matrix transform:

Theorem 1: If

$f(X) \in L_{P_i}(0,\infty); i = 1,2,\ldots,r\ 1 \leq P_i \leq 2[or f(X) \in M_{P_i}(0,\infty)\ and\ P_i > 2]$ where

$\mathrm{Re}(\alpha + \min\limits_{1\leq j\leq M} \frac{b_j}{\beta_j}) > \frac{m+1}{2}, \mathrm{Re}(\sigma) > -\frac{1}{Q_i}, \mathrm{Re}(t) > \frac{m+1}{2}, \mathrm{Re}(\sigma - t + 1) > \frac{m+1}{2}$ and

$|\arg(I-a)| < \pi$ then

$$M\{Y[f(X)]\} = \frac{\Gamma_m\left(\sigma - t + \frac{m+1}{2}\right)}{\Gamma_m(\rho)} I_{P_i+1,Q_i+1:r}^{M,N+1}\left[\gamma I \Bigg|_{B^*,\left(\frac{m+1}{2}-t-\sigma-\rho,1\right)}^{\left(\frac{m+1}{2},1\right),A^*}\right] M[f(U)] \ (3.2.1)$$

Where I is $m \times m$ identity matrix.

Proof: Taking the matrix transform of equation (3.1.7), we get

$$M\{Y[f(X)]\} = \int_{X>0} |X|^{t-\frac{m+1}{2}} \left[\frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)} \right.$$

$$\left. \int_{0<U<X} |U|^{\sigma} |X-U|^{\rho-\frac{m+1}{2}} I_{P_i,Q_i:r}^{M,N}\left[\gamma(I-UX^{-1})\Big|_{B^*}^{A^*}\right] f(U)dU \right] dX$$

Changing the order of integration which is permissible under the conditions stated with the theorem, we obtain

$$M\{Y[f(X)]\} = \frac{1}{\Gamma_m(\rho)} \int_{X>0} |U|^{\sigma} \; f(U)dU$$

$$\left. \int_{0<U<X} |X|^{t-\sigma-\rho-\frac{m+1}{2}} |X-U|^{\rho-\frac{m+1}{2}} I_{P_i,Q_i:r}^{M,N}\left[\gamma(I-UX^{-1})\Big|_{B^*}^{A^*}\right] \right] dX$$

On evaluating X -integral with the help of the result given by Mathai and Saxena (1978),

$$\int_0^1 |X|^{\delta-\frac{m+1}{2}} |I-X|^{\rho-\frac{m+1}{2}} H_{r,s}^{p,q}\left[|XZ| \Big|_{(b_j,\beta_j)_{1,s}}^{(a_j,\alpha_j)_{1,r}} \right] dX =$$

$$\Gamma_m(\rho) H_{r+1,s+1}^{p,q+1}\left[|Z| \Big|_{(b_j,\beta_j)_{1,s},\left(\frac{m+1}{2}-\delta-\rho,1\right)}^{\left(\frac{m+1}{2}-\delta,1\right),(a_j,\alpha_j)_{1,r}} \right] \qquad (3.2.2)$$

Where $\mathrm{Re}(\alpha + \min_{1\le j\le M} \frac{b_j}{\beta_j}) > \frac{m+1}{2}$ and $\mathrm{Re}(\rho) > \frac{m+1}{2} - 1$.

We obtain the required result.

Theorem 2: If $f(X) \in L_{P_i}(0,\infty)$ $1 \le P_i \le 2[or f(X) \in M_{P_i}(0,\infty)\, and\, P_i > 2]$ where

$\mathrm{Re}(\delta + \min_{1\le j\le M} \frac{b_j}{\beta_j}) > \frac{m+1}{2}, \mathrm{Re}(\delta) > -\frac{1}{Q_i}, \mathrm{Re}(t) > \frac{m+1}{2}, \mathrm{Re}(\delta+t) > \frac{m+1}{2}$ and

$|\arg(I-a)| < \pi$ then

$$M\{N[f(X)]\} = \frac{\Gamma_m(\delta+1)}{\Gamma_m(\rho)} I_{P_i+1,Q_i+1}^{M,N+1}\left[\gamma I \Big|_{B^*,\left(\frac{m+1}{2}-\delta-\rho,1\right)}^{\left(\frac{m+1}{2}-\delta,1\right),A^*} \right] M[f(U)] \qquad (3.2.3)$$

Where I is $m \times m$ identity matrix.

Proof: Taking the matrix transform of equation (3.2.2), we get

$$M\left\{N[f(X)]\right\}=\int_{X>0}|X|^{\delta-\frac{m+1}{2}}\left[\frac{|X|^{\delta}}{\Gamma_m(\rho)}\right.$$

$$\left.\int_{U>X}|U|^{-\sigma-\rho}|U-X|^{\rho-\frac{m+1}{2}}I_{P_i,Q_i;r}^{M,N}\left[\gamma(I-XU^{-1})\Big|_{B^*}^{A^*}\right]f(U)dU\right]dX$$

And changing the order of integration and evaluation X -integral with the help of (3.2.2), we obtain the required result.

Theorem 3: If $f(X)\in L_{P_i}(0,\infty), g(X)\in L_{P_i}(0,\infty)$ where

$$\mathrm{Re}(\delta+\min_{1\le j\le M}\frac{b_j}{\beta_j})>\frac{m+1}{2},\mathrm{Re}(\delta)>-\frac{1}{Q_i},\mathrm{Re}(\rho)>\frac{m+1}{2},\mathrm{Re}(\sigma)>\max\left(\frac{1}{P_i}+\frac{1}{Q_i}\right)\text{and}$$

$|\arg(I-a)|<\pi$ then

$$\int_{X>0}f(X)Y\left[g(X)\middle|\sigma,\rho,\gamma;_{B^*}^{A^*}\right]dX=$$

$$\int_{X>0}g(X)N\left[f(X)\middle|\sigma,\rho,\gamma;_{B^*}^{A^*}\right]dX \qquad (3.2.4)$$

Proof: Equation (3.2.4) immediately follows on interpreting it with the help of equations (3.1.7) and (3.1.8).

Special Cases

(i) If we put $M=1, N=1, P_i=2, Q_i=2, r=1, \gamma=1$, then the operators (3.1.7) and (3.1.8) reduce to their Mellin transforms in the following form:

$$Y[f(X)]=$$

$$\frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)}\int_{0<U<X}|U|^{\sigma}|X-U|^{\rho-\frac{m+1}{2}}H_{2,2}^{1,2}\left[(I-UX^{-1})\right]f(U)dU$$

Here $Y[f(X)]=Y\left[f(X)\middle|\sigma,\rho,1;_{(b_1,\beta_1),(b_2,\beta_2)_{1,2}}^{(a_1;\alpha_1),(a_2;\alpha_2)_{1,2}}\right]$

And $H_{2,2}^{1,2}\left[(I-UX^{-1})\right]=H_{2,2}^{1,2}\left[(I-UX^{-1})\Big|_{(b_1,\beta_1),(b_2,\beta_2)}^{(a_1,\alpha_1),(a_2,\alpha_2)}\right]$

Then

$$Y[f(X)]=\frac{\Gamma_m(\chi_1)|X|^{-\sigma-\frac{m+1}{2}}}{\Gamma_m(\rho)}$$

$$\int_{0<U<X}|U|^{\sigma}|I-UX^{-1}|^{\rho-\beta_1-\frac{m+1}{2}}{}_2F_1\left[-;(I-UX^{-1})\right]f(U)dU$$

Where $\Gamma(\chi_1)=\dfrac{\Gamma_m\left(\frac{m+1}{2}-\alpha_1+\beta_1\right)\Gamma_m\left(\frac{m+1}{2}-\alpha_2+\beta_2\right)}{\Gamma_m\left(\frac{m+1}{2}-\alpha_1-\alpha_2+\beta_1+\beta_2\right)}$

$$ {}_2F_1\left[-;\left(I-UX^{-1}\right)\right]= $$
$$ {}_2F_1\left[\frac{m+1}{2}-\alpha_1-\beta_1,\frac{m+1}{2}-\alpha_2-\beta_2;\frac{m+1}{2}-\beta_1-\beta_2;-(I-UX^{-1})\right] $$

By virtue of the result Srivastava, Gupta and Goyal (1982).

Taking M -transform on both sides, we get

$$ M\left\{Y(f(X)\right\}=\frac{\Gamma_m(\chi_1)\Gamma_m(\chi_2)}{\Gamma_m(\rho)}\,{}_3F_2(-;I)M[f(U)] $$

Where $\Gamma(\chi_2)=\dfrac{\Gamma_m\left(\frac{m+1}{2}+\sigma\right)\Gamma_m\left(\rho+\beta_1\right)}{\Gamma_m\left(\sigma+\rho+\beta_1+\frac{m+1}{2}\right)}$

And

$$ {}_3F_2(-;I)={}_3F_2\left(\frac{m+1}{2}-\alpha_1+\beta_1,\frac{m+1}{2}-\alpha_2+\beta_2,\sigma+\frac{m+1}{2};\right. $$
$$ \left.\frac{m+1}{2}-\beta_2+\beta_1,\sigma+\rho+\frac{m+1}{2}+\beta_1;I\right), $$

$$ N[f(X)]=\frac{|X|^{\delta}}{\Gamma_m(\rho)}\int_{U>X}|U|^{\sigma}|U-X|^{\rho-\frac{m+1}{2}}H_{2,2}^{1,2}\left[(I-XU^{-1})\right]f(U)dU $$

Where $\quad N[f(X)]=N\left[f(X)\middle|\sigma,\rho,1;{}^{(a_1;\alpha_1),(a_2;\alpha_2)_{1,2}}_{(b_1,\beta_1),(b_2,\beta_2)_{1,2}}\right]$

And $\quad H_{2,2}^{1,2}\left[(I-XU^{-1})\right]=H_{2,2}^{1,2}\left[(I-XU^{-1})\middle|^{(a_1,\alpha_1),(a_2,\alpha_2)}_{(b_1,\beta_1),(b_2,\beta_2)}\right]$

Then $\quad N[f(X)]=\dfrac{\Gamma_m(\chi_1)|X|^{\delta+\rho-\frac{m+1}{2}}}{\Gamma_m(\rho)}$

$$ \int_{0<U<X}|U|^{-\delta-\rho}|I-XU^{-1}|^{\rho+\beta_1-\frac{m+1}{2}}\,{}_2F_1\left[-;(I-XU^{-1})\right]f(U)dU $$

Where

$$ {}_2F_1\left[-;\left(I-XU^{-1}\right)\right]= $$

$$ {}_2F_1\left[\frac{m+1}{2}-\alpha_1+\beta_1,\frac{m+1}{2}-\alpha_2+\beta_2;\frac{m+1}{2}+\beta_1-\beta_2;-(I-XU^{-1})\right] $$

Taking M -transform on both sides, we get

$$ M\left\{N(f(X)\right\}=\frac{(-1)^{\rho-\frac{m+1}{2}}\Gamma_m(\chi_1)\Gamma_m(\chi_2)}{\Gamma_m(\rho)}\,{}_2F_1(-;I)M[f(U)] $$

(ii) Putting $P_i=0,Q_1=1,M=1,N=0,\gamma=1,r=1$, then operators (3.1.7) and (3.1.8)

reduce to their Mellin transform in the following forms:

$$ Y[f(X)]=\frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)} $$

$$ \int\limits_{0<U<X}|U|^{\sigma}|X-U|^{\rho-\frac{m+1}{2}}H_{0,1}^{1,0}\left[(I-UX^{-1})\Big|_{(\beta,1)}^{-}\right]f(U)dU $$

Where $\quad Y[f(X)]=Y\left[f(X)\Big|\sigma,\rho,1;_{(\beta,1)}^{-}\right]$

And

$$ H_{0,1}^{1,0}\left[(I-UX^{-1})\Big|_{(\beta,1)}^{-}\right]=|I-UX^{-1}|^{\beta}\,e^{-tr(i-UX^{-1})} $$

$$ =\frac{|X|^{-\sigma-\frac{m+1}{2}}}{\Gamma_m(\rho)}\int\limits_{0<U<X}|U|^{\sigma}|I-UX^{-1}|^{\rho+\beta-\frac{m+1}{2}}e^{-tr(1-UX^{-1})}f(U)dU $$

By virtue of the result (1982). Taking M -transform on both sides, we get

$$ M\left\{Y(f(X)\right\}=\frac{\Gamma_m(\rho+\beta)}{\Gamma_m(\rho)}M[f(U)] $$

Also $\quad N[f(X)]=\dfrac{|X|^{\delta}}{\Gamma_m(\rho)}$

$$ \int\limits_{U>X}|U|^{-\delta-\rho}|U-X|^{\rho-\frac{m+1}{2}}H_{0,1}^{1,0}\left[(I-XU^{-1})\Big|_{(\beta,1)}^{-}\right]f(U)dU $$

Where $\quad N[f(X)]=N\left[f(X)\Big|\delta,\rho,1;_{(\beta,1)}^{-}\right]$

And

$$H_{0,1}^{1,0}\left[(I-XU^{-1})\Big|_{(\beta,1)}^{-}\right]=|I-XU^{-1}|^{\beta}\, e^{-tr(i-XU^{-1})}$$

$$=\frac{|X|^{\delta}}{\Gamma_m(\rho)}\int_{U>X}|U|^{-\delta-\frac{m+1}{2}}|I-XU^{-1}|^{\rho+\beta-\frac{m+1}{2}}\, e^{-tr(1-XU^{-1})}f(U)dU$$

By virtue of the result (1995). Taking M -transform on both sides, we get

$$M\{N[f(X)]\}=\frac{\Gamma_m\left(\rho-\delta+\beta-\frac{m+1}{2}\right)}{\Gamma_m(\rho)}M[f(U)]$$

If we put $\alpha_j=\beta_j=1;(j=1,\ldots,P;\ j=1,\ldots,Q)$ the operators reduce to G -function given

by Vyas (1996).

Theorem4. If $f(X)\in L_{P_i}(0,\infty)\ 1\le P_i\le 2[or f(X)\in M_{P_i}(0,\infty)\ and\ P_i>2]$ where

$\mathrm{Re}(\alpha)>\frac{m+1}{2},\mathrm{Re}(\sigma)>-\frac{1}{Q_i},\mathrm{Re}(\rho)>\frac{m+1}{2},\mathrm{Re}(\alpha-\sigma)>\frac{m+1}{2}$ and

$|\arg(I-a)|<\pi$ then

$$M\{R[_{\sigma,\rho,a}^{\alpha};f(X)]\}=\frac{\Gamma_m\left(\sigma-s+\frac{m+1}{2}\right)\Gamma_m(\rho+\alpha)}{\Gamma_m\left(\sigma-s+\alpha+\rho+\frac{m+1}{2}\right)\Gamma_m(\rho)}$$

$${}_1F_1\left[\rho+\alpha;\sigma-s+\alpha+\rho+\frac{m+1}{2};I\right]M[f(U)] \qquad (3.2.5)$$

Proof: Using the Mellin transform of

$$R[f(X)]=R\left[_{\sigma,\rho,a}^{(a_r),(b_r)};f(X)\right]=$$

$$\frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)}\int_{0<U<X}|U|^{\sigma}|X-U|^{\rho-\frac{m+1}{2}}\,G_{r,s}^{p,q}\left[a(I-UX^{-1})\Big|_{(b_r)}^{(a_r)}\right]f(U)dU \qquad (3.2.6)$$

We get

$$M\{R[_{\sigma,\rho,a}^{\alpha};f(X)]\}=\int_{X>0}\frac{|X|^{s-\frac{m+1}{2}}|X|^{-\sigma-\rho}}{\Gamma_m(\rho)}$$

$$\left[\int_{0<U<X}|U|^{\sigma}|X-U|^{\rho-\frac{m+1}{2}}\,G_{0,1}^{1,0}\left[a(I-UX^{-1})\Big|_{(b_1)}^{-}\right]f(U)dU\right]dX$$

Changing the order of integration which is permissible under the conditions stated with the theorem, we obtain

$$\int\limits_{0<U<X} |X|^{s-\sigma-\rho-\frac{m+1}{2}} |X-U|^{\rho-\frac{m+1}{2}} G_{0,1}^{1,0}\left[a(I-UX^{-1})\right]\Bigg] dX =$$

$$\frac{1}{\Gamma_m(\rho)} \int\limits_{U>X} |U|^{\sigma} f(U)dU$$

$$\int\limits_{X>U} |X|^{S-\sigma-\rho-\alpha-\frac{m+1}{2}} |X-U|^{\rho+\alpha-\frac{m+1}{2}} e^{-tr(1-UX^{-1})} dX$$

On evaluating X -integral with the help of result given by Mathai (1995)

$$\int_0^1 e^{-tr(XZ)} |X|^{\delta-\frac{m+1}{2}} |I-X|^{\rho-\delta-\frac{m+1}{2}} dX$$

$$= \frac{\Gamma_m(\delta)\Gamma_m(\rho-\delta)}{\Gamma_m(\rho)} {}_1F_1[\delta;\rho;-Z] \qquad (3.2.7)$$

For $\mathrm{Re}(\delta) > \frac{m+1}{2}, \mathrm{Re}(\rho) > \frac{m+1}{2}, \mathrm{Re}(\rho-\delta) > \frac{m+1}{2}$

We obtain the required result.

Theorem5. If $f(X) \in L_{P_i}(0,\infty)\ 1 \le P_i \le 2[or f(X) \in M_{P_i}(0,\infty)\ and\ P_i > 2]$ where

$$\mathrm{Re}(\alpha) > \frac{m+1}{2}, \mathrm{Re}(\delta) > -\frac{1}{P_i}, \mathrm{Re}(\rho) > \frac{m+1}{2}, \mathrm{Re}(\alpha-\rho) > \frac{m+1}{2}, \frac{1}{P_i} + \frac{1}{Q_i} = 1 \text{ and}$$

$|\arg(I-a)| < \pi$ then

$$M\left\{K[^{\alpha}_{\delta,\rho,a}; f(X)]\right\} = \frac{\Gamma_m\left(\delta+s+\frac{m+1}{2}\right)\Gamma_m(\rho+\alpha)}{\Gamma_m\left(\delta+s+\alpha+\rho+\frac{m+1}{2}\right)\Gamma_m(\rho)}$$

$${}_1F_1\left[\rho+\alpha;\delta+s+\alpha+\rho+\frac{m+1}{2};I\right] M[f(U)] \qquad (3.2.8)$$

Proof: Using the Mellin transform of

$$K[f(X)] = K\left[^{(a_r),(b_r)}_{\sigma,\rho,a}; f(X)\right] = \frac{|X|^{\delta}}{\Gamma_m(\rho)}$$

$$\int\limits_{U>X} |U|^{\delta} |X-U|^{\rho-\frac{m+1}{2}} G_{r,s}^{p,q}\left[a(I-XU^{-1})\Big|^{(a_r)}_{(b_r)}\right] f(U)dU \qquad (3.2.9)$$

We get

$$M\left\{K[^{\alpha}_{\delta,\rho,a}; f(X)]\right\} = \int_{X>0} \frac{|X|^{s-\frac{m+1}{2}}|X|^{\delta}}{\Gamma_m(\rho)}$$

$$\left[\int_{U>X} |U|^{-\delta-\rho}|U-X|^{\rho-\frac{m+1}{2}} \; G^{1,0}_{0,1}\left[a(I-XU^{-1})\Big|^{-}_{(b_1)}\right] f(U)dU\right]dX \qquad (3.2.10)$$

Changing the order of integration and evaluating X -integral with the help of (3.2.5), we obtain the required result.

When $M=1, N=0, P=0, Q=1, a=1$ in (3.2.6) and (3.2.9) reduces to the following from of operators:

$$R[f(X)] = R\left[^{\alpha,\beta}_{\sigma,\rho,1}; f(X)\right] = \frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)}$$

$$\int_{0<U<X} |U|^{\sigma}|X-U|^{\rho-\frac{m+1}{2}} \; G^{1,0}_{0,1}\left[a(I-UX^{-1})\Big|^{\alpha}_{\beta}\right] f(U)dU \qquad (3.2.11)$$

And $\quad K[f(X)] = K\left[^{\alpha,\beta}_{\sigma,\rho,1}; f(X)\right] = \frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)}$

$$\int_{U>X} |U|^{-\delta-\sigma}|U-X|^{\rho-\frac{m+1}{2}} \; G^{1,0}_{0,1}\left[a(I-UX^{-1})\Big|^{\alpha}_{\beta}\right] f(U)dU \qquad (3.2.12)$$

Theorem6. If $f(X) \in L_{P_i}(0,\infty)\ 1 \le P_i \le 2[or f(X) \in M_{P_i}(0,\infty)\ and\ P_i > 2]$ where

$\mathrm{Re}(\alpha) > \frac{m+1}{2}, \mathrm{Re}(\sigma) > -\frac{1}{Q_i}, \mathrm{Re}(\rho) > \frac{m+1}{2}, \mathrm{Re}(\alpha-\sigma) > \frac{m+1}{2}, \frac{1}{P_i} + \frac{1}{Q_i} = 1$ and

$|\arg(I-a)| < \pi$ then

$$M\left\{R[^{\alpha}_{\sigma,\rho,1}; f(X)]\right\} =$$

$$\frac{\Gamma_m(\sigma+\alpha-\beta-s)\Gamma_m(\rho+\beta)}{\Gamma_m(\sigma-s+\alpha+\rho)\Gamma_m(\rho)\Gamma_m(\alpha-\beta)} M[f(U)] \qquad (3.2.13)$$

Proof: Using the Mellin transform of (3.2.11), we get

$$M\left\{R[^{\alpha}_{\sigma,\rho,1}; f(X)]\right\} = \int_{X>0} \frac{|X|^{-\sigma-\rho}|X|^{s-\frac{m+1}{2}}}{\Gamma_m(\rho)}$$

$$\left[\int_{0<U<X} |U|^{\alpha}|X-U|^{\rho-\frac{m+1}{2}} G_{1,1}^{1,0}\left[a(I-UX^{-1})\Big|_{\beta}^{\alpha}\right] f(U)dU\right]dX$$

Changing the order of integration which is permissible under the conditions stated with

the theorem, we obtain

$$M\left\{R[_{\sigma,\rho,1}^{\alpha}; f(X)]\right\} = \frac{1}{\Gamma_m(\rho)} \int_{U>0} |U|^{\sigma} f(U)dU$$

$$\int G_{1,1}^{1,0}\left[(I-XU^{-1})\Big|_{\beta}^{\alpha}\right] |X|^{s-\sigma-\rho-\frac{m+1}{2}} |X-U|^{\rho-\frac{m+1}{2}} dX$$

Using the result given by Mathai (1995).

$$G_{1,1}^{1,0}\left[X\Big|_{\beta}^{\alpha}\right] = \frac{1}{\Gamma_m(\alpha-\beta)} |X|^{\beta} |I-U|^{\alpha-\beta-\frac{m+1}{2}} \tag{3.2.14}$$

Provided $0 < X < I, \mathrm{Re}(\alpha-\beta) > \frac{m+1}{2}$

We get $M\left\{R[_{\sigma,\rho,1}^{\alpha,\beta}; f(X)]\right\} = \frac{1}{\Gamma_m(\rho)} \int_{U>0} |U|^{\sigma-\alpha-\beta-\frac{m+1}{2}} f(U)dU$

$$\frac{1}{\Gamma_m(\alpha-\beta)} \int_{X>U} |X|^{s-\sigma-\alpha} |X-U|^{\beta+\rho-\frac{m+1}{2}} dX$$

On evaluating X -integral with the help of the following result

$$\int_0^1 |X|^{\delta-\frac{m+1}{2}} |I-X|^{\rho-\frac{m+1}{2}} dX = \frac{\Gamma_m(\delta)\Gamma_m(\rho)}{\Gamma_m(\rho+\delta)} \tag{3.2.15}$$

For $\mathrm{Re}(\delta) > 0, \mathrm{Re}(\rho) > \frac{m+1}{2}$

We arrive at the required result.

In the present paper fractional integral operators associated with A-function for real positive symmetric definite matrix have been discussed. These operators have a wide range of applications in the field of Mathematical Physics and Linear differential equations. A number of special cases of our operators have been mentioned.

Introduction

Fractional integration is an immediate generalization of repeated integration. Fractional integral operators occur in the solutions of linear differential equations, partial differential equations and in the integral representations of hypergeometric functions of one or more variables. Riesz and Garding respectively introduced Riemann-Liouville integral of vector and matrix variables and applied them in the solution of differential equation associated with Cauchy's problem.

A-function with matrix argument

Let X is a $p\times p$ real symmetric positive definite matrix of functionally independent variables. Let the A-function of X be denoted by

${}_1(a_j,\alpha_j)_n$ Represents the set of n pairs of parameters the A-function was defined by Gautam G.P. and Goyal as

$$A_{p,q}^{m,n}\left[x\left|\begin{matrix}{}_1(a_j,\alpha_j)_p\\ {}_1(b_j,\beta_j)_q\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_L f(s)x^s ds \qquad (3.3.1)$$

Where

$$f(s)=\frac{\prod_{j=1}^{m}\Gamma(a_j+\alpha_j s)\prod_{j=1}^{n}\Gamma(1-b_j-\beta_j s)}{\prod_{j=m+1}^{p}\Gamma(1-a_j-\alpha_j s)\prod_{j=n+1}^{q}\Gamma(b_j+\beta_j s)} \qquad (3.3.2)$$

The integral on the right hand side of (3.3.1) is convergent when $f>0$ and $|\arg(ux)|<\frac{f\pi}{2}$, where

$$f=\mathrm{Re}(\sum_{j=1}^{m}\alpha_j-\sum_{j=m+1}^{p}\alpha_j+\sum_{j=1}^{n}\beta_j-\sum_{j=n+1}^{q}\beta_j\,,\quad u=\prod_{j=1}^{p}\alpha_j^{\alpha_j}\prod_{j=1}^{q}\beta_j^{-\beta_j} \qquad (3.3.3)$$

(3.3.1) reduces to H-function given by Fox the following relation

$$A_{p,q}^{n,m}\left[x\left|\begin{matrix}{}_1(1-a_j,\alpha_j)_p\\ {}_1(1-b_j,\beta_j)_q\end{matrix}\right.\right]=H_{p,q}^{m,n}\left[x\left|\begin{matrix}{}_1(a_j,\alpha_j)_p\\ {}_1(b_j,\beta_j)_q\end{matrix}\right.\right] \qquad (3.3.4)$$

It is assumed that $A(XY)=A(YX)$ for real symmetric $m\times m$ positive definite matrices X and Y, $A(X)$ is defined by the following integral equation:

$$\int_{X>0}|X|^{\rho-\frac{m+1}{2}}A(X)dX=\frac{\prod_{j=1}^{m}\Gamma_m(1-b_j-\beta_j\xi)\prod_{j=1}^{n}\Gamma_m(1-\frac{m+1}{2}+a_j+\alpha_j\xi)}{\left\{\prod_{j=m+1}^{q_i}\Gamma_m(1-\frac{m+1}{2}+b_{ji}+\beta_{ji}\xi)\prod_{j=n+1}^{p_i}\Gamma_m(1-a_j-\alpha_{ji}\xi)\right\}} \qquad (3.3.5)$$

Sethi discussed the following fractional integral operators involving H-function of matrix arguments:

$$R[f(X)]=R\left[{}^{(a_j,\alpha_j)_{1,r},(b_j,\beta_j)_{1,s}}_{\sigma,\rho,\gamma};f(X)\right]$$

$$=\frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)}\int_{0<U<X}|U|^{\sigma}|X-U|^{\rho-\frac{m+1}{2}}H_{r,s}^{p,q}\left[\gamma(I-UX^{-1})\left|\begin{matrix}(a_j,\alpha_j)_{1,r}\\(b_j,\beta_j)_{1,s}\end{matrix}\right.\right]f(U)dU \qquad (3.3.6)$$

$$K[f(X)]=K\left[{}^{(a_j,\alpha_j)_{1,r},(b_j,\beta_j)_{1,s}}_{\delta,\rho,\gamma};f(X)\right]=$$

$$\frac{|X|^{\delta}}{\Gamma_m(\rho)}\int_{U>X}|U|^{-\delta-\rho}|U-X|^{\rho-\frac{m+1}{2}}H_{r,s}^{p,q}\left[\gamma(I-XU^{-1})\left|\begin{matrix}(a_j,\alpha_j)_{1,r}\\(b_j,\beta_j)_{1,s}\end{matrix}\right.\right]f(U)dU \qquad (3.3.7)$$

Where $f(X) = f(x_{11}, \ldots, x_{1m}, x_{21}, \ldots, x_{mm})$ be a real bounded function of a complex parameter.

(1.2) Matrix transform

A generalized matrix transform or M-transform of a function $f(X)$ of a $m \times m$ real symmetric positive definite or strictly negative definite matrix X is defined as follows:

$$M_f(s) = \int_{X>0} |X|^{s-\frac{m+1}{2}} f(X)dX \qquad (X > 0) \qquad (3.3.9)$$

Whenever $M_f(s)$ exists. Also $f(X)$ is assumed to be a symmetric function i.e.

$f(BX) = f(XB) = f\left(B^{1/2} X B^{1/2}\right)$ for $B = B' > 0$. When $X < 0$ replace X by $-X$ in M-transform.

(1.3) Integral operators involving A-function

$$Y[f(X)] = Y\left[f(X) \middle| \sigma, \rho, \gamma; {}^{1(a_j,\alpha_j)_p}_{1(b_j,\beta_j)_q} \right] =$$

$$\frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)} \int_{0<U<X} |U|^{\sigma} |X-U|^{\rho-\frac{m+1}{2}} A^{M,N}_{P,Q}\left[\gamma(1-UX^{-1}) \middle|^{1(a_j,\alpha_j)_p}_{1(b_j,\beta_j)_q} \right] f(U)dU \qquad (3.3.10)$$

$$N[f(X)] = N\left[f(X) \middle| \delta, \rho, \gamma; {}^{1(a_j,\alpha_j)_p}_{1(b_j,\beta_j)_q} \right] =$$

$$\frac{|X|^{\delta}}{\Gamma_m(\rho)} \int_{U>X} |U|^{-\delta-\rho} |U-X|^{\rho-\frac{m+1}{2}} A^{M,N}_{P,Q}\left[\gamma(I-XU^{-1}) \middle|^{1(a_j,\alpha_j)_p}_{1(b_j,\beta_j)_q} \right] f(U)dU \qquad (3.3.8)$$

The above defined operators exists under the following conditions:

(i) $P \geq 1, Q < \infty, \frac{1}{P} + \frac{1}{Q} = 1, |\arg(I-1+a)| < \pi$

(ii) ($\mathrm{Re}(\sigma) > \frac{1}{Q}, \mathrm{Re}(\delta) > \frac{1}{P}, \mathrm{Re}(\rho) > \frac{m+1}{2}$

(iii) $\mathrm{Re}(\beta + \min_{1\leq j\leq M} \frac{1-a_j}{\alpha_j}) > \frac{m+1}{2}$, (iv) $f(X) \in L_P(0,\infty)$.

The last condition ensures that $Y[f(X)]$ and $N[f(X)]$ both exist and also both belong to $L_P(0,\infty)$.

Main Results

The following theorems of the operators defined by (3.3.9) and (3.3.10) have been established in the expression of matrix transform:

Theorem 1: If $f(X) \in L_P(0,\infty)$ $1 \leq P \leq 2[or f(X) \in M_P(0,\infty)\, and\, P > 2]$ where $\mathrm{Re}(\beta + \min_{1\leq j\leq M} \frac{1-a_j}{\alpha_j}) > \frac{m+1}{2}, \mathrm{Re}(\sigma) > -\frac{1}{Q}, \mathrm{Re}(t) > \frac{m+1}{2}, \mathrm{Re}(\sigma - t + 1) > \frac{m+1}{2}$ and $|\arg(I-1+a)| < \pi$ then

$$M\left\{Y[f(X)]\right\}=\frac{\Gamma_m\left(\sigma-t+\frac{m+1}{2}\right)}{\Gamma_m(\rho)}A^{M,N+1}_{P+1,Q+1}\left[\gamma I\left|^{\left(1-\frac{m+1}{2},1\right),_1(a_j,\alpha_j)_p}_{_1(b_j,\beta_j)_q,\left(1-\frac{m+1}{2}+t+\sigma+\rho,1\right)}\right.\right]M[f(U)] \qquad (3.4.1)$$

Where I is $m\times m$ identity matrix.

Proof: Taking the matrix transform of equation (3.3.9), we get

$$M\left\{Y[f(X)]\right\}=\int_{X>0}|X|^{t-\frac{m+1}{2}}\left[\frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)}\right.$$

$$\left.\int_{0<U<X}|U|^{\sigma}|X-U|^{\rho-\frac{m+1}{2}}A^{M,N}_{P,Q}\left[\gamma(I-UX^{-1})\left|^{_1(a_j,\alpha_j)_q}_{_1(b_j,\beta_j)_q}\right.\right]f(U)dU\right]dX$$

Changing the order of integration which is permissible under the conditions stated with the theorem, we obtain

$$M\left\{Y[f(X)]\right\}=\frac{1}{\Gamma_m(\rho)}\int_{X>0}|U|^{\sigma}\ f(U)dU$$

$$\left.\left.\int_{0<U<X}|X|^{t-\sigma-\rho-\frac{m+1}{2}}|X-U|^{\rho-\frac{m+1}{2}}A^{M,N}_{P,Q}\left[\gamma(I-UX^{-1})\left|^{_1(a_j,\alpha_j)_q}_{_1(b_j,\beta_j)_q}\right.\right]\right]\right]dX$$

On evaluating X -integral with the help of the result given by Mathai and Saxena,

$$\int_0^1|X|^{\delta-\frac{m+1}{2}}|I-X|^{\rho-\frac{m+1}{2}}H^{p,q}_{r,s}\left[|XZ|\left\|^{(a_j,\alpha_j)_{1,r}}_{(b_j,\beta_j)_{1,s}}\right.\right]dX=$$

$$\Gamma_m(\rho)H^{p,q+1}_{r+1,s+1}\left[|Z|\left\|^{\left(\frac{m+1}{2}-\delta,1\right),(a_j,\alpha_j)_{1,r}}_{(b_j,\beta_j)_{1,s},\left(\frac{m+1}{2}-\delta-\rho,1\right)}\right.\right] \qquad (3.4.2)$$

Where $\mathrm{Re}(\alpha+\min_{1\le j\le M}\frac{b_j}{\beta_j})>\frac{m+1}{2}$ and $\mathrm{Re}(\rho)>\frac{m+1}{2}-1$.

We obtain the required result.

Theorem 2: If $f(X)\in L_P(0,\infty)\ 1\le P\le 2[or\ f(X)\in M_P(0,\infty)\ and\ P>2]$ where $\mathrm{Re}(\delta+\min_{1\le j\le M}\frac{1-a_j}{\alpha_j})>\frac{m+1}{2},\mathrm{Re}(\delta)>-\frac{1}{Q},\mathrm{Re}(t)>\frac{m+1}{2},\mathrm{Re}(\delta+t)>\frac{m+1}{2}$ and $|\arg(I-a)|<\pi$ then

$$M\left\{N[f(X)]\right\}=\frac{\Gamma_m(\delta+1)}{\Gamma_m(\rho)}A^{M,N+1}_{P+1,Q+1}\left[\gamma I\left|^{\left(1-\frac{m+1}{2}+\delta,1\right),_1(a_j,\alpha_j)_p}_{_1(b_j,\beta_j)_q,\left(1-\frac{m+1}{2}+\delta+\rho,1\right)}\right.\right]M[f(U)] \qquad (3.4.3)$$

Where I is $m\times m$ identity matrix.

Proof: Taking the matrix transform of equation (3.4.2), we get

$$M\left\{N[f(X)]\right\} = \int_{X>0} |X|^{\delta-\frac{m+1}{2}} \left[\frac{|X|^{\delta}}{\Gamma_m(\rho)} \right.$$

$$\left. \int_{U>X} |U|^{-\sigma-\rho} |U-X|^{\rho-\frac{m+1}{2}} I_{P,Q}^{M,N}\left[\gamma(I-XU^{-1})\Big|_{{}_1(b_j,\beta_j)_q}^{{}_1(a_j,\alpha_j)_q}\right] f(U)dU \right] dX$$

And changing the order of integration and evaluation X -integral with the help of (3.4.2), we obtain the required result.

Theorem 3: If $f(X)\in L_P(0,\infty), g(X)\in L_P(0,\infty)$ where

$\mathrm{Re}(\delta+\min_{1\le j\le M}\frac{1-a_j}{\alpha_j})>\frac{m+1}{2}, \mathrm{Re}(\delta)>-\frac{1}{Q}, \mathrm{Re}(\rho)>\frac{m+1}{2}, \mathrm{Re}(\sigma)>\max\left(\frac{1}{P}+\frac{1}{Q}\right)$ and $|\arg(I-a)|<\pi$ then

$$\int_{X>0} f(X)Y\left[g(X)\middle|\sigma,\rho,\gamma;{}_{{}_1(b_j,\beta_j)_q}^{{}_1(a_j,\alpha_j)_p}\right]dX = \int_{X>0} g(X)N\left[f(X)\middle|\sigma,\rho,\gamma;{}_{{}_1(b_j,\beta_j)_q}^{{}_1(a_j,\alpha_j)_p}\right]dX \qquad (3.4.4)$$

Proof: Equation (3.4.4) immediately follows on interpreting it with the help of equations (3.3.9) and (3.3.10).

Special Cases

(i) If we put $M=1, N=1, P=2, Q=2, a_j=1-a_j; b_j=1-b_j, \gamma=1$,then the operators (3.3.9) and (3.3.10) reduce to their Mellin transforms in the following form:

$$Y[f(X)] = \frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)} \int_{0<U<X} |U|^{\sigma} |X-U|^{\rho-\frac{m+1}{2}} H_{2,2}^{1,2}\left[(I-UX^{-1})\right] f(U)dU$$

Here $\quad Y[f(X)] = Y\left[f(X)\middle|\sigma,\rho,1;{}_{(b_1,\beta_1),(b_2,\beta_2)}^{(a_1;\alpha_1),(a_2;\alpha_2)}\right]$

And $\quad H_{2,2}^{1,2}\left[(I-UX^{-1})\right] = H_{2,2}^{1,2}\left[(I-UX^{-1})\Big|_{(b_1,\beta_1),(b_2,\beta_2)}^{(a_1,\alpha_1),(a_2,\alpha_2)}\right]$

Then

$$Y[f(X)] = \frac{\Gamma_m(\chi_1)|X|^{-\sigma-\frac{m+1}{2}}}{\Gamma_m(\rho)} \int_{0<U<X} |U|^{\sigma} |I-UX^{-1}|^{\rho-\beta_1-\frac{m+1}{2}} {}_2F_1\left[-;(I-UX^{-1})\right] f(U)dU$$

Where

$$\Gamma(\chi_1) = \frac{\Gamma_m\left(\frac{m+1}{2}-\alpha_1+\beta_1\right)\Gamma_m\left(\frac{m+1}{2}-\alpha_2+\beta_2\right)}{\Gamma_m\left(\frac{m+1}{2}-\alpha_1-\alpha_2+\beta_1+\beta_2\right)}$$

$${}_2F_1\left[-;\left(I-UX^{-1}\right)\right] =$$

$${}_2F_1\left[\frac{m+1}{2}-\alpha_1-\beta_1, \frac{m+1}{2}-\alpha_2-\beta_2; \frac{m+1}{2}-\beta_1-\beta_2; -(I-UX^{-1})\right]$$

By virtue of the result .

Taking M -transform on both sides, we get

$$M\{Y(f(X)\}=\frac{\Gamma_m(\chi_1)\Gamma_m(\chi_2)}{\Gamma_m(\rho)}\,{}_3F_2(-;I)M[f(U)]$$

Where $\Gamma(\chi_2)=\dfrac{\Gamma_m\left(\frac{m+1}{2}+\sigma\right)\Gamma_m\left(\rho+\beta_1\right)}{\Gamma_m\left(\sigma+\rho+\beta_1+\frac{m+1}{2}\right)}$

And

$${}_3F_2(-;I)={}_3F_2\left(\frac{m+1}{2}-\alpha_1+\beta_1,\frac{m+1}{2}-\alpha_2+\beta_2,\sigma+\frac{m+1}{2};\right.$$
$$\left.\frac{m+1}{2}-\beta_2+\beta_1,\sigma+\rho+\frac{m+1}{2}+\beta_1;I\right),$$

$$N[f(X)]=\frac{|X|^{\delta}}{\Gamma_m(\rho)}\int_{U>X}|U|^{\sigma}|U-X|^{\rho-\frac{m+1}{2}}H_{2,2}^{1,2}\left[(I-XU^{-1})\right]f(U)dU$$

Where $N[f(X)]=N\left[f(X)\middle|\sigma,\rho,1;_{(b_1,\beta_1),(b_2,\beta_2)}^{(a_1;\alpha_1),(a_2;\alpha_2)}\right]$

And $H_{2,2}^{1,2}\left[(I-XU^{-1})\right]=H_{2,2}^{1,2}\left[(I-XU^{-1})\middle|_{(b_1,\beta_1),(b_2,\beta_2)}^{(a_1,\alpha_1),(a_2,\alpha_2)}\right]$

Then

$$N[f(X)]=\frac{\Gamma_m(\chi_1)|X|^{\delta+\rho-\frac{m+1}{2}}}{\Gamma_m(\rho)}\int_{0<U<X}|U|^{-\delta-\rho}|I-XU^{-1}|^{\rho+\beta_1-\frac{m+1}{2}}\,{}_2F_1\left[-;(I-XU^{-1})\right]f(U)dU$$

Where

$${}_2F_1\left[-;\left(I-XU^{-1}\right)\right]=$$
$${}_2F_1\left[\frac{m+1}{2}-\alpha_1+\beta_1,\frac{m+1}{2}-\alpha_2+\beta_2;\frac{m+1}{2}+\beta_1-\beta_2;-(I-XU^{-1})\right]$$

Taking M -transform on both sides, we get

$$M\{N(f(X)\}=\frac{(-1)^{\rho-\frac{m+1}{2}}\Gamma_m(\chi_1)\Gamma_m(\chi_2)}{\Gamma_m(\rho)}\,{}_2F_1(-;I)M[f(U)]$$

(ii) Putting $P=0,Q=1,M=1,N=0,\gamma=1,a_j=1-a_j,b_j=1-b_j$, then operators (3.3.9) and (3.3.10) reduce to their Mellin transform in the following forms:

$$Y[f(X)] = \frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)} \int_{0<U<X} |U|^{\sigma} |X-U|^{\rho-\frac{m+1}{2}} H_{0,1}^{1,0}\left[(I-UX^{-1})\Big|_{(\beta,1)}^{-}\right] f(U)dU$$

Where $Y[f(X)] = Y\left[f(X)\middle|\sigma,\rho,1;_{(\beta,1)}^{-}\right]$

And $H_{0,1}^{1,0}\left[(I-UX^{-1})\Big|_{(\beta,1)}^{-}\right] = |I-UX^{-1}|^{\beta} e^{-tr(i-UX^{-1})}$

$$= \frac{|X|^{-\sigma-\frac{m+1}{2}}}{\Gamma_m(\rho)} \int_{0<U<X} |U|^{\sigma} |I-UX^{-1}|^{\rho+\beta-\frac{m+1}{2}} e^{-tr(1-UX^{-1})} f(U)dU$$

By virtue of the result . Taking M -transform on both sides, we get

$$M\left\{Y(f(X)\right\} = \frac{\Gamma_m(\rho+\beta)}{\Gamma_m(\rho)} M[f(U)]$$

Also

$$N[f(X)] = \frac{|X|^{\delta}}{\Gamma_m(\rho)} \int_{U>X} |U|^{-\delta-\rho} |U-X|^{\rho-\frac{m+1}{2}} H_{0,1}^{1,0}\left[(I-XU^{-1})\Big|_{(\beta,1)}^{-}\right] f(U)dU$$

Where $N[f(X)] = N\left[f(X)\middle|\delta,\rho,1;_{(\beta,1)}^{-}\right]$

And $H_{0,1}^{1,0}\left[(I-XU^{-1})\Big|_{(\beta,1)}^{-}\right] = |I-XU^{-1}|^{\beta} e^{-tr(i-XU^{-1})}$

$$= \frac{|X|^{\delta}}{\Gamma_m(\rho)} \int_{U>X} |U|^{-\delta-\frac{m+1}{2}} |I-XU^{-1}|^{\rho+\beta-\frac{m+1}{2}} e^{-tr(1-XU^{-1})} f(U)dU$$

Taking M -transform on both sides, we get

$$M\left\{N[f(X)]\right\} = \frac{\Gamma_m\left(\rho-\delta+\beta-\frac{m+1}{2}\right)}{\Gamma_m(\rho)} M[f(U)]$$

If we put $\alpha_j = \beta_j = 1; (j=1,\ldots,P;\ j=1,\ldots,Q)$ the operators reduce to G -function given by Vyas .

Theorem4. If $f(X) \in L_P(0,\infty)\ 1 \le P \le 2[or f(X) \in M_P(0,\infty)\ and\ P>2]$ where $\operatorname{Re}(\alpha) > \frac{m+1}{2}, \operatorname{Re}(\sigma) > -\frac{1}{Q}, \operatorname{Re}(\rho) > \frac{m+1}{2}, \operatorname{Re}(\alpha-\sigma) > \frac{m+1}{2}$ and $|\arg(I-a)| < \pi$ then

$$M\left\{R[_{\sigma,\rho,a}^{\alpha}; f(X)]\right\} = \frac{\Gamma_m\left(\sigma-s+\frac{m+1}{2}\right)\Gamma_m(\rho+\alpha)}{\Gamma_m\left(\sigma-s+\alpha+\rho+\frac{m+1}{2}\right)\Gamma_m(\rho)}$$

$$ {}_1F_1\left[\rho+\alpha;\sigma-s+\alpha+\rho+\frac{m+1}{2};I\right]M[f(U)] \quad (3.3.5)$$

Proof: Using the Mellin transform of

$$R[f(X)] = R\left[{}^{(a_r),(b_r)}_{\sigma,\rho,a}; f(X)\right] =$$

$$\frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)} \int_{0<U<X} |U|^{\sigma} |X-U|^{\rho-\frac{m+1}{2}} G^{p,q}_{r,s}\left[a(I-UX^{-1})\Big|^{(a_r)}_{(b_r)}\right] f(U)dU \quad (3.3.6)$$

We get

$$M\left\{R[^{\alpha}_{\sigma,\rho,a}; f(X)]\right\} =$$

$$\int_{X>0} \frac{|X|^{s-\frac{m+1}{2}}|X|^{-\sigma-\rho}}{\Gamma_m(\rho)}\left[\int_{0<U<X} |U|^{\sigma}|X-U|^{\rho-\frac{m+1}{2}} G^{1,0}_{0,1}\left[a(I-UX^{-1})\Big|^{-}_{(b_1)}\right] f(U)dU\right]dX$$

Changing the order of integration which is permissible under the conditions stated with the theorem, we obtain

$$\int_{0<U<X} |X|^{s-\sigma-\rho-\frac{m+1}{2}}|X-U|^{\rho-\frac{m+1}{2}} G^{1,0}_{0,1}\left[a(I-UX^{-1})\right]\Bigg]dX = \frac{1}{\Gamma_m(\rho)}\int_{U>X} |U|^{\sigma}\; f(U)dU$$

$$\int_{X>U} |X|^{S-\sigma-\rho-\alpha-\frac{m+1}{2}}|X-U|^{\rho+\alpha-\frac{m+1}{2}} e^{-tr(1-UX^{-1})}dX$$

On evaluating X -integral with the help of result given by Mathai

$$\int_0^1 e^{-tr(XZ)}|X|^{\delta-\frac{m+1}{2}}|I-X|^{\rho-\delta-\frac{m+1}{2}}\,dX = \frac{\Gamma_m(\delta)\Gamma_m(\rho-\delta)}{\Gamma_m(\rho)}\,{}_1F_1[\delta;\rho;-Z] \quad (3.3.7)$$

For $\mathrm{Re}(\delta) > \frac{m+1}{2}, \mathrm{Re}(\rho) > \frac{m+1}{2}, \mathrm{Re}(\rho-\delta) > \frac{m+1}{2}$

We obtain the required result.

Theorem5. If $f(X) \in L_p(0,\infty)\; 1 \le P \le 2[or\ f(X) \in M_p(0,\infty)\, and\ P > 2]$ where $\mathrm{Re}(\alpha) > \frac{m+1}{2}, \mathrm{Re}(\delta) > -\frac{1}{P}, \mathrm{Re}(\rho) > \frac{m+1}{2}, \mathrm{Re}(\alpha-\rho) > \frac{m+1}{2}, \frac{1}{P}+\frac{1}{Q} = 1$ and $|\arg(I-a)| < \pi$ then

$$M\left\{K[^{\alpha}_{\delta,\rho,a}; f(X)]\right\} = \frac{\Gamma_m\left(\delta+s+\frac{m+1}{2}\right)\Gamma_m(\rho+\alpha)}{\Gamma_m\left(\delta+s+\alpha+\rho+\frac{m+1}{2}\right)\Gamma_m(\rho)}$$

$$ {}_1F_1\left[\rho+\alpha;\delta+s+\alpha+\rho+\frac{m+1}{2};I\right]M[f(U)] \quad (3.3.8)$$

Proof: Using the Mellin transform of

$$K[f(X)] = K\left[{}^{(a_r),(b_r)}_{\sigma,\rho,a}; f(X)\right] =$$

$$\frac{|X|^{\delta}}{\Gamma_m(\rho)} \int_{U>X} |U|^{\delta} |X-U|^{\rho-\frac{m+1}{2}} G^{p,q}_{r,s}\left[a(I-XU^{-1})\Big|^{(a_r)}_{(b_r)}\right] f(U)dU \qquad (3.3.9)$$

We get $\quad M\left\{K[^{\alpha}_{\delta,\rho,a}; f(X)]\right\} =$

$$\int_{X>0} \frac{|X|^{s-\frac{m+1}{2}} |X|^{\delta}}{\Gamma_m(\rho)} \left[\int_{U>X} |U|^{-\delta-\rho} |U-X|^{\rho-\frac{m+1}{2}} G^{1,0}_{0,1}\left[a(I-XU^{-1})\Big|^{-}_{(b_1)}\right] f(U)dU\right] dX \quad (3.3.10)$$

Changing the order of integration and evaluating X -integral with the help of (3.3.5), we obtain the required result.

When $M=1, N=0, P=0, Q=1, a=1$ in (3.3.6) and (3.3.9) reduces to the following from of operators:

$$R[f(X)] = R\left[{}^{\alpha,\beta}_{\sigma,\rho,1}; f(X)\right] =$$

$$\frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)} \int_{0<U<X} |U|^{\sigma} |X-U|^{\rho-\frac{m+1}{2}} G^{1,0}_{0,1}\left[a(I-UX^{-1})\Big|^{\alpha}_{\beta}\right] f(U)dU \qquad (3.3.11)$$

And $\quad K[f(X)] = K\left[{}^{\alpha,\beta}_{\sigma,\rho,1}; f(X)\right] =$

$$\frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)} \int_{U>X} |U|^{-\delta-\sigma} |U-X|^{\rho-\frac{m+1}{2}} G^{1,0}_{0,1}\left[a(I-UX^{-1})\Big|^{\alpha}_{\beta}\right] f(U)dU \qquad (3.3.12)$$

Theorem6. If $f(X) \in L_P(0,\infty)$ $1 \le P \le 2[or\ f(X) \in M_P(0,\infty)\ and\ P > 2]$ where

$\mathrm{Re}(\alpha) > \frac{m+1}{2}, \mathrm{Re}(\sigma) > -\frac{1}{Q}, \mathrm{Re}(\rho) > \frac{m+1}{2}, \mathrm{Re}(\alpha-\sigma) > \frac{m+1}{2}, \frac{1}{P} + \frac{1}{Q} = 1$ and $|\arg(I-a)| < \pi$ then

$$M\left\{R[^{\alpha}_{\sigma,\rho,1}; f(X)]\right\} = \frac{\Gamma_m(\sigma+\alpha-\beta-s)\Gamma_m(\rho+\beta)}{\Gamma_m(\sigma-s+\alpha+\rho)\Gamma_m(\rho)\Gamma_m(\alpha-\beta)} M[f(U)] \qquad (3.3.13)$$

Proof: Using the Mellin transform of (3.3.11), we get

$$M\left\{R[^{\alpha}_{\sigma,\rho,1}; f(X)]\right\} =$$

$$\int_{X>0} \frac{|X|^{-\sigma-\rho} |X|^{s-\frac{m+1}{2}}}{\Gamma_m(\rho)} \left[\int_{0<U<X} |U|^{\alpha} |X-U|^{\rho-\frac{m+1}{2}} G^{1,0}_{1,1}\left[a(I-UX^{-1})\Big|^{\alpha}_{\beta}\right] f(U)dU\right] dX$$

Changing the order of integration which is permissible under the conditions stated with the theorem, we obtain

$$M\left\{R[^{\alpha}_{\sigma,\rho,1}; f(X)]\right\} = \frac{1}{\Gamma_m(\rho)} \int_{U>0} |U|^{\sigma} f(U)dU$$

$$\int G^{1,0}_{1,1}\left[(I - XU^{-1})\Big|^{\alpha}_{\beta}\right] |X|^{s-\sigma-\rho-\frac{m+1}{2}} |X-U|^{\rho-\frac{m+1}{2}} dX$$

Using the result given by Mathai .

$$G^{1,0}_{1,1}\left[X\Big|^{\alpha}_{\beta}\right] = \frac{1}{\Gamma_m(\alpha-\beta)} |X|^{\beta} |I-U|^{\alpha-\beta-\frac{m+1}{2}} \tag{3.3.14}$$

Provided $\quad 0 < X < I, \mathrm{Re}(\alpha-\beta) > \dfrac{m+1}{2}$

We get $\quad M\left\{R[^{\alpha,\beta}_{\sigma,\rho,1}; f(X)]\right\} = \dfrac{1}{\Gamma_m(\rho)} \displaystyle\int_{U>0} |U|^{\sigma-\alpha-\beta-\frac{m+1}{2}} f(U)dU$

$$\frac{1}{\Gamma_m(\alpha-\beta)} \int_{X>U} |X|^{s-\sigma-\alpha} |X-U|^{\beta+\rho-\frac{m+1}{2}} dX$$

On evaluating X -integral with the help of the following result

$$\int_0^1 |X|^{\delta-\frac{m+1}{2}} |I-X|^{\rho-\frac{m+1}{2}} dX = \frac{\Gamma_m(\delta)\Gamma_m(\rho)}{\Gamma_m(\rho+\delta)} \tag{3.3.15}$$

For $\mathrm{Re}(\delta) > 0, \mathrm{Re}(\rho) > \dfrac{m+1}{2}$

We arrive at the required result.

Introduction

Fractional integration is an immediate generalization of repeated integration. Fractional integral operators occur in the solutions of linear differential equations, partial differential equations and in the integral representations of hypergeometric functions of one or more variables. Riesz and Garding respectively introduced Riemann-Liouville integral of vector and matrix variables and applied them in the solution of differential equation associated with Cauchy's problem.

(a) $\overline{H}$ -function with matrix argument

Let X is a $p \times p$ real symmetric positive definite matrix of functionally independent variables. Let the $\overline{H}$ -function of X be denoted by

$$\overline{H}^{M,N}_{P,Q}[z] = \overline{H}^{M,N}_{P,Q}\left[z\Big|^{(a_j;\alpha_j;A_j)_{1,N},(a_j;\alpha_j)_{N+1,P}}_{(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}}\right] = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \overline{\phi}(\xi) z^{\xi} d\xi \tag{3.4.1}$$

where $$\bar{\phi}(\xi)=\frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\left\{\Gamma(1-a_j+\alpha_j\xi)\right\}^{A_j}}{\prod_{j=M+1}^{Q}\left\{\Gamma(1-b_j+\beta_j\xi)\right\}^{B_j}\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j\xi)} \qquad (3.4.2)$$

Which contains fractional powers of the gamma functions. Here, and throughout the paper $a_j(j=1,\ldots,p)$ and $b_j(j=1,\ldots,Q)$ are complex parameters, $\alpha_j \geq 0(j=1,\ldots,P), \beta_j \geq 0(j=1,\ldots,Q)$ (not all zero simultaneously) and exponents $A_j(j=1,\ldots,N)$ and $B_j(j=N+1,\ldots,Q)$ can take on non integer values.

The following sufficient condition for the absolute convergence of the defining integral for the $\overline{H}$-function given by equation (3.4.1) have been given by (Buschman and Srivastava[1]).

$$\Omega \equiv \sum_{j=1}^{M}\left|\beta_j\right|+\sum_{j=1}^{N}\left|A_j\alpha_j\right|-\sum_{j=M+1}^{Q}\left|\beta_jB_j\right|-\sum_{j=N+1}^{P}\left|\alpha_j\right|>0 \qquad (3.4.3)$$

and $\left|\arg(z)\right|<\frac{1}{2}\pi\,\Omega$ (3.4.4)

The behavior of the $\overline{H}$-function for small values of $|z|$ follows easily from a result recently given by (Rathie ,p.306,eq.(6.9)).

We have

$$\overline{H}_{P,Q}^{M,N}\left[z\right]=0\left(\left|z\right|^{\gamma}\right),\gamma=\min_{1\leq j\leq N}\left[\operatorname{Re}\left(b_j/\beta_j\right)\right],\left|z\right|\to 0 \qquad (3.4.5)$$

If we take $A_j=1\left(j=1,\ldots,N\right)$, $B_j=1\left(j=M+1,\ldots,Q\right)$ in (1.1), the function $\overline{H}_{P,Q}^{M,N}$ reduces to the Fox's H-function [2].

It is assumed that $\overline{H}(XY)=\overline{H}(YX)$ for real symmetric $m\times m$ positive definite matrices X and Y, $\overline{H}(X)$ is defined by the following integral equation:

$$\int_{X>0}|X|^{\rho-\frac{m+1}{2}}\,\overline{H}(X)dX=\frac{\prod_{j=1}^{M}\Gamma_m(b_j-\beta_j\xi)\prod_{j=1}^{N}\left\{\Gamma_m(\frac{m+1}{2}-a_j+\alpha_j\xi)\right\}^{A_j}}{\prod_{j=M+1}^{Q}\left\{\Gamma_m(\frac{m+1}{2}-b_j+\beta_j\xi)\right\}^{B_j}\prod_{j=N+1}^{P}\Gamma_m(a_j-\alpha_j\xi)} \qquad (3.4.6)$$

Sethi discussed the following fractional integral operators involving H-function of matrix arguments:

$$R[f(X)]=R\left[{}^{(a_j,\alpha_j)_{1,r},(b_j,\beta_j)_{1,s}}_{\sigma,\rho,\gamma};f(X)\right]$$

$$=\frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)}\int_{0<U<X}|U|^{\sigma}|X-U|^{\rho-\frac{m+1}{2}}H_{r,s}^{p,q}\left[\gamma(I-UX^{-1})\Big|_{(b_j,\beta_j)_{1,s}}^{(a_j,\alpha_j)_{1,r}}\right]f(U)dU \qquad (3.4.7)$$

$$K[f(X)]=K\left[{}^{(a_j,\alpha_j)_{1,r},(b_j,\beta_j)_{1,s}}_{\delta,\rho,\gamma};f(X)\right]=$$

$$\frac{|X|^{\delta}}{\Gamma_m(\rho)}\int_{U>X}|U|^{-\delta-\rho}|U-X|^{\rho-\frac{m+1}{2}}H_{r,s}^{p,q}\left[\gamma(I-XU^{-1})\Big|_{(b_j,\beta_j)_{1,s}}^{(a_j,\alpha_j)_{1,r}}\right]f(U)dU \qquad (3.4.8)$$

Where $f(X)=f(x_{11},\ldots,x_{1m},x_{21},\ldots,x_{mm})$ be a real bounded function of a complex parameter.

(b) Matrix transform

A generalized matrix transform or M-transform of a function $f(X)$ of a $m\times m$ real symmetric positive definite or strictly negative definite matrix X is defined as follows:

$$M_f(s)=\int_{X>0}|X|^{s-\frac{m+1}{2}}f(X)dX \qquad (X>0) \qquad (3.4.9)$$

Whenever $M_f(s)$ exists. Also $f(X)$ is assumed to be a symmetric function i.e.

$f(BX)=f(XB)=f\left(B^{1/2}XB^{1/2}\right)$ for $B=B'>0$. When $X<0$ replace X by $-X$ in M -transform.

(c) Integral operators involving $\overline{H}$ -function

$$Y[f(X)]=Y\left[f(X)\big|\sigma,\rho,\gamma;_{(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}}^{(a_j;\alpha_j;A_j)_{1,N},(a_j;\alpha_j)_{N+1,P}}\right]=$$

$$\frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)}\int_{0<U<X}|U|^{\sigma}|X-U|^{\rho-\frac{m+1}{2}}\overline{H}_{P,Q}^{M,N}\left[\gamma(1-UX^{-1})\Big|_{(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}}^{(a_j;\alpha_j;A_j)_{1,N},(a_j;\alpha_j)_{N+1,P}}\right]f(U)dU \quad (3.4.10)$$

$$N[f(X)]=N\left[f(X)\big|\delta,\rho,\gamma;_{(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}}^{(a_j;\alpha_j;A_j)_{1,N},(a_j;\alpha_j)_{N+1,P}}\right]=$$

$$\frac{|X|^{\delta}}{\Gamma_m(\rho)}\int_{U>X}|U|^{-\delta-\rho}|U-X|^{\rho-\frac{m+1}{2}}\overline{H}_{P,Q}^{M,N}\left[\gamma(I-XU^{-1})\Big|_{(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}}^{(a_j;\alpha_j;A_j)_{1,N},(a_j;\alpha_j)_{N+1,P}}\right]f(U)dU$$
(3.4.11)

The above defined operators exists under the following conditions:

(i) $P\geq 1, Q<\infty, \frac{1}{P}+\frac{1}{Q}=1, |\arg(I-a)|<\pi$

(ii) $(\operatorname{Re}(\sigma)>\frac{1}{Q}, \operatorname{Re}(\delta)>\frac{1}{P}, \operatorname{Re}(\rho)>\frac{m+1}{2}$

(iii) $\operatorname{Re}(\alpha+\min_{1\leq j\leq M}\frac{b_j}{\beta_j})>\frac{m+1}{2}$, (iv) $f(X)\in L_P(0,\infty)$.

The last condition ensures that $Y[f(X)]$ and $N[f(X)]$ both exist and also both belong to $L_P(0,\infty)$.

Main Results

The following theorems of the operators defined by (3.4.10) and (3.4.11) have been established in the expression of matrix transform:

Theorem 1: If $f(X) \in L_P(0,\infty)\ 1 \le P \le 2[or f(X) \in M_P(0,\infty)\ and\ P > 2]$ where

$$\mathrm{Re}(\alpha + \min_{1\le j\le M} \frac{b_j}{\beta_j}) > \frac{m+1}{2}, \mathrm{Re}(\sigma) > -\frac{1}{Q}, \mathrm{Re}(t) > \frac{m+1}{2}, \mathrm{Re}(\sigma - t + 1) > \frac{m+1}{2} \text{ and } |\arg(I-a)| < \pi \text{ then}$$

$$M\{Y[f(X)]\} = \frac{\Gamma_m\left(\sigma - t + \frac{m+1}{2}\right)}{\Gamma_m(\rho)}$$

$$\overline{H}^{M,N+1}_{P+1,Q+1}\left[\gamma I\left|\begin{matrix}\left(\frac{m+1}{2},1;1\right),(a_j;\alpha_j;A_j)_{1,N},(a_j;\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},\left(\frac{m+1}{2}-t-\sigma-\rho,1;1\right)\end{matrix}\right.\right] M[f(U)] \tag{3.5.1}$$

Where I is $m \times m$ identity matrix.

Proof: Taking the matrix transform of equation (3.4.10), we get

$$M\{Y[f(X)]\} = \int_{X>0} |X|^{t-\frac{m+1}{2}} \left[\frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)}\right.$$

$$\left.\int_{0<U<X} |U|^{\sigma} |X-U|^{\rho-\frac{m+1}{2}} \overline{H}^{M,N}_{P,Q}\left[\gamma(I-UX^{-1})\left|\begin{matrix}(a_j;\alpha_j;A_j)_{1,N},(a_j;\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right] f(U)dU\right]dX$$

Changing the order of integration which is permissible under the conditions stated with the theorem, we obtain

$$M\{Y[f(X)]\} = \frac{1}{\Gamma_m(\rho)} \int_{X>0} |U|^{\sigma} f(U)dU$$

$$\int_{0<U<X} |X|^{t-\sigma-\rho-\frac{m+1}{2}} |X-U|^{\rho-\frac{m+1}{2}} \overline{H}^{M,N}_{P,Q}\left[\gamma(I-UX^{-1})\left|\begin{matrix}(a_j;\alpha_j;A_j)_{1,N},(a_j;\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]dX$$

On evaluating X -integral with the help of the result given by Mathai and Saxena ,

$$\int_0^1 |X|^{\delta-\frac{m+1}{2}} |I-X|^{\rho-\frac{m+1}{2}} H^{p,q}_{r,s}\left[|XZ|\left\|\begin{matrix}(a_j,\alpha_j)_{1,r}\\(b_j,\beta_j)_{1,s}\end{matrix}\right.\right]dX =$$

$$\Gamma_m(\rho) H^{p,q+1}_{r+1,s+1}\left[|Z|\left\|\begin{matrix}\left(\frac{m+1}{2}-\delta,1\right),(a_j,\alpha_j)_{1,r}\\(b_j,\beta_j)_{1,s},\left(\frac{m+1}{2}-\delta-\rho,1\right)\end{matrix}\right.\right]$$

Where $\mathrm{Re}(\alpha + \min_{1\le j\le M} \frac{b_j}{\beta_j}) > \frac{m+1}{2}$ and $\mathrm{Re}(\rho) > \frac{m+1}{2} - 1$.

We obtain the required result.

Theorem 2: If $f(X) \in L_P(0,\infty)$ $1 \le P \le 2[or f(X) \in M_P(0,\infty)\, and\; P > 2]$ where

$\mathrm{Re}(\delta + \min_{1\le j\le M} \frac{b_j}{\beta_j}) > \frac{m+1}{2}, \mathrm{Re}(\delta) > -\frac{1}{Q}, \mathrm{Re}(t) > \frac{m+1}{2}, \mathrm{Re}(\delta + t) > \frac{m+1}{2}$ and $|\arg(I-a)| < \pi$ then

$$M\{N[f(X)]\} = \frac{\Gamma_m(\delta+1)}{\Gamma_m(\rho)}$$

$$\overline{H}^{M,N+1}_{P+1,Q+1}\left[\gamma I \left| \begin{matrix} \left(\frac{m+1}{2}-\delta,1;1\right),(a_j;\alpha_j;A_j)_{1,N},(a_j;\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},\left(\frac{m+1}{2}-\delta-\rho,1;1\right) \end{matrix} \right.\right] M[f(U)] \qquad (3.5.2)$$

Where I is $m\times m$ identity matrix.

Proof: Taking the matrix transform of equation (3.4.11), we get

$$M\{N[f(X)]\} = \int_{X>0} |X|^{\delta-\frac{m+1}{2}} \left[\frac{|X|^{\delta}}{\Gamma_m(\rho)} \right.$$

$$\left. \int_{U>X} |U|^{-\sigma-\rho} |U-X|^{\rho-\frac{m+1}{2}} \overline{H}^{M,N}_{P,Q}\left[\gamma(I-XU^{-1})\left| \begin{matrix} (a_j;\alpha_j;A_j)_{1,N},(a_j;\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q} \end{matrix} \right.\right] f(U)dU \right] dX \qquad (3.5.3)$$

And changing the order of integration and evaluation X -integral with the help of (3.5.3), we obtain the required result.

Theorem 3: If $f(X) \in L_P(0,\infty), g(X) \in L_P(0,\infty)$ where

$\mathrm{Re}(\delta + \min_{1\le j\le M} \frac{b_j}{\beta_j}) > \frac{m+1}{2}, \mathrm{Re}(\delta) > -\frac{1}{Q}, \mathrm{Re}(\rho) > \frac{m+1}{2}, \mathrm{Re}(\sigma) > \max\left(\frac{1}{P}+\frac{1}{Q}\right)$ and $|\arg(I-a)| < \pi$ then

$$\int_{X>0} f(X) Y\left[g(X) \middle| \sigma,\rho,\gamma; \begin{matrix} (a_j;\alpha_j;A_j)_{1,N},(a_j;\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q} \end{matrix} \right] dX =$$

$$\int_{X>0} g(X) N\left[f(X) \middle| \sigma,\rho,\gamma; \begin{matrix} (a_j;\alpha_j;A_j)_{1,N},(a_j;\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q} \end{matrix} \right] dX \qquad (3.5.4)$$

Proof: Equation (3.5.4) immediately follows on interpreting it with the help of equations (3.4.10) and (3.4.11).

Special Cases

(i) If we put $M=1, N=1, P=2, Q=2, A_j=1=B_j, \gamma=1$, then the operators (3.4.10) and (3.4.11) reduce to their Mellin transforms in the following form:

$$Y[f(X)] = \frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)} \int_{0<U<X} |U|^{\sigma} |X-U|^{\rho-\frac{m+1}{2}} H^{1,2}_{2,2}\left[(I-UX^{-1})\right] f(U)dU$$

Here $Y[f(X)] = Y\left[f(X) \middle| \sigma, \rho, 1;^{(a_1;\alpha_1;1),(a_2;\alpha_2)_{1,2}}_{(b_1,\beta_1),(b_2,\beta_2;1)_{1,2}} \right]$

And

$$H^{1,2}_{2,2}\left[(I-UX^{-1})\right] = H^{1,2}_{2,2}\left[(I-UX^{-1})\Big|^{(a_1,\alpha_1;1),(a_2,\alpha_2)}_{(b_1,\beta_1),(b_2,\beta_2;1)}\right]$$

Then

$$Y[f(X)] = \frac{\Gamma_m(\chi_1)|X|^{-\sigma-\frac{m+1}{2}}}{\Gamma_m(\rho)} \int\limits_{0<U<X} |U|^{\sigma}|I-UX^{-1}|^{\rho-\beta_1-\frac{m+1}{2}} {}_2F_1\left[-;(I-UX^{-1})\right] f(U)dU$$

Where $\Gamma(\chi_1) = \dfrac{\Gamma_m\left(\frac{m+1}{2}-\alpha_1+\beta_1\right)\Gamma_m\left(\frac{m+1}{2}-\alpha_2+\beta_2\right)}{\Gamma_m\left(\frac{m+1}{2}-\alpha_1-\alpha_2+\beta_1+\beta_2\right)}$

$$\begin{aligned}&{}_2F_1\left[-;\left(I-UX^{-1}\right)\right] = \\ &{}_2F_1\left[\frac{m+1}{2}-\alpha_1-\beta_1, \frac{m+1}{2}-\alpha_2-\beta_2; \frac{m+1}{2}-\beta_1-\beta_2; -(I-UX^{-1})\right]\end{aligned}$$

By virtue of the result.

Taking M -transform on both sides, we get

$$M\left\{Y(f(X)\right\} = \frac{\Gamma_m(\chi_1)\Gamma_m(\chi_2)}{\Gamma_m(\rho)} {}_3F_2(-;I)M[f(U)]$$

Where $\Gamma(\chi_2) = \dfrac{\Gamma_m\left(\frac{m+1}{2}+\sigma\right)\Gamma_m\left(\rho+\beta_1\right)}{\Gamma_m\left(\sigma+\rho+\beta_1+\frac{m+1}{2}\right)}$

And

$$\begin{aligned}{}_3F_2(-;I) = {}_3F_2\Bigg(&\frac{m+1}{2}-\alpha_1+\beta_1, \frac{m+1}{2}-\alpha_2+\beta_2, \sigma+\frac{m+1}{2}; \\ &\frac{m+1}{2}-\beta_2+\beta_1, \sigma+\rho+\frac{m+1}{2}+\beta_1; I\Bigg),\end{aligned}$$

$$N[f(X)] = \frac{|X|^{\delta}}{\Gamma_m(\rho)} \int\limits_{U>X} |U|^{\sigma}|U-X|^{\rho-\frac{m+1}{2}} H^{1,2}_{2,2}\left[(I-XU^{-1})\right] f(U)dU$$

Where $N[f(X)] = N\left[f(X) \middle| \sigma, \rho, 1;^{(a_1;\alpha_1;1),(a_2;\alpha_2)_{1,2}}_{(b_1,\beta_1),(b_2,\beta_2;1)_{1,2}} \right]$

And $H_{2,2}^{1,2}\left[(I-XU^{-1})\right]=H_{2,2}^{1,2}\left[(I-XU^{-1})\Big|_{(b_1,\beta_1),(b_2,\beta_2;1)}^{(a_1,\alpha_1;1),(a_2,\alpha_2)}\right]$

Then

$$N[f(X)]=\frac{\Gamma_m(\chi_1)|X|^{\delta+\rho-\frac{m+1}{2}}}{\Gamma_m(\rho)}\int_{0<U<X}|U|^{-\delta-\rho}|I-XU^{-1}|^{\rho+\beta_1-\frac{m+1}{2}}\,{}_2F_1\left[-;(I-XU^{-1})\right]f(U)dU$$

Where

$${}_2F_1\left[-;\left(I-XU^{-1}\right)\right]=$$
$${}_2F_1\left[\frac{m+1}{2}-\alpha_1+\beta_1,\frac{m+1}{2}-\alpha_2+\beta_2;\frac{m+1}{2}+\beta_1-\beta_2;-(I-XU^{-1})\right]$$

Taking M -transform on both sides, we get

$$M\left\{N(f(X)\right\}=\frac{(-1)^{\rho-\frac{m+1}{2}}\Gamma_m(\chi_1)\Gamma_m(\chi_2)}{\Gamma_m(\rho)}\,{}_2F_1(-;I)M[f(U)]$$

(ii) Putting $P=0,Q=1,M=1,N=0,\gamma=1,A_j=1=B_j$, then operators (3.4.10) and (3.4.11) reduce to their Mellin transform in the following forms:

$$Y[f(X)]=\frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)}\int_{0<U<X}|U|^{\sigma}|X-U|^{\rho-\frac{m+1}{2}}H_{0,1}^{1,0}\left[(I-UX^{-1})\Big|_{(\beta,1)}^{-}\right]f(U)dU$$

Where $\quad Y[f(X)]=Y\left[f(X)\middle|\sigma,\rho,1;_{(\beta,1)}^{-}\right]$

And $\quad H_{0,1}^{1,0}\left[(I-UX^{-1})\Big|_{(\beta,1)}^{-}\right]=|I-UX^{-1}|^{\beta}e^{-tr(i-UX^{-1})}$

$$=\frac{|X|^{-\sigma-\frac{m+1}{2}}}{\Gamma_m(\rho)}\int_{0<U<X}|U|^{\sigma}|I-UX^{-1}|^{\rho+\beta-\frac{m+1}{2}}e^{-tr(1-UX^{-1})}f(U)dU$$

Taking M -transform on both sides, we get

$$M\left\{Y(f(X)\right\}=\frac{\Gamma_m(\rho+\beta)}{\Gamma_m(\rho)}M[f(U)]$$

Also

$$N[f(X)]=\frac{|X|^{\delta}}{\Gamma_m(\rho)}\int_{U>X}|U|^{-\delta-\rho}|U-X|^{\rho-\frac{m+1}{2}}H_{0,1}^{1,0}\left[(I-XU^{-1})\Big|_{(\beta,1)}^{-}\right]f(U)dU$$

Where $\qquad N[f(X)]=N\left[f(X)\middle|\delta,\rho,1;_{(\beta,1)}^{-}\right]$

And $H_{0,1}^{1,0}\left[(I-XU^{-1})\Big|_{(\beta,1)}^{-}\right]=|I-XU^{-1}|^{\beta}\,e^{-tr(i-XU^{-1})}$

$$=\frac{|X|^{\delta}}{\Gamma_m(\rho)}\int_{U>X}|U|^{-\delta-\frac{m+1}{2}}|I-XU^{-1}|^{\rho+\beta-\frac{m+1}{2}}\,e^{-tr(1-XU^{-1})}f(U)dU$$

Taking M -transform on both sides, we get

$$M\left\{N[f(X)]\right\}=\frac{\Gamma_m\left(\rho-\delta+\beta-\frac{m+1}{2}\right)}{\Gamma_m(\rho)}M[f(U)]$$

If we put $\alpha_j=\beta_j=1;(j=1,\ldots,P;\ j=1,\ldots,Q)$ the operators reduce to G -function given by Vyas .

Theorem4. If $f(X)\in L_P(0,\infty)\ 1\le P\le 2[or f(X)\in M_P(0,\infty)\ and\ P>2]$ where $\mathrm{Re}(\alpha)>\frac{m+1}{2},\mathrm{Re}(\sigma)>-\frac{1}{Q},\mathrm{Re}(\rho)>\frac{m+1}{2},\mathrm{Re}(\alpha-\sigma)>\frac{m+1}{2}$ and $|\arg(I-a)|<\pi$ then

$$M\left\{R[_{\sigma,\rho,a}^{\alpha};f(X)]\right\}=\frac{\Gamma_m\left(\sigma-s+\frac{m+1}{2}\right)\Gamma_m(\rho+\alpha)}{\Gamma_m\left(\sigma-s+\alpha+\rho+\frac{m+1}{2}\right)\Gamma_m(\rho)}$$

$${}_1F_1\left[\rho+\alpha;\sigma-s+\alpha+\rho+\frac{m+1}{2};I\right]M[f(U)] \qquad (3.5.5)$$

Proof: Using the Mellin transform of

$R[f(X)]=R\left[{}_{\sigma,\rho,a}^{(a_r),(b_r)};f(X)\right]=$

$$\frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)}\int_{0<U<X}|U|^{\sigma}|X-U|^{\rho-\frac{m+1}{2}}G_{r,s}^{p,q}\left[a(I-UX^{-1})\Big|_{(b_r)}^{(a_r)}\right]f(U)dU \qquad (3.5.6)$$

We get $\quad M\left\{R[_{\sigma,\rho,a}^{\alpha};f(X)]\right\}=$

$$\int_{X>0}\frac{|X|^{s-\frac{m+1}{2}}|X|^{-\sigma-\rho}}{\Gamma_m(\rho)}\left[\int_{0<U<X}|U|^{\sigma}|X-U|^{\rho-\frac{m+1}{2}}G_{0,1}^{1,0}\left[a(I-UX^{-1})\Big|_{(b_1)}^{-}\right]f(U)dU\right]dX$$

Changing the order of integration which is permissible under the conditions stated with the theorem, we obtain

$$\int_{0<U<X}|X|^{s-\sigma-\rho-\frac{m+1}{2}}|X-U|^{\rho-\frac{m+1}{2}}G_{0,1}^{1,0}\left[a(I-UX^{-1})\right]\Bigg]dX=\frac{1}{\Gamma_m(\rho)}\int_{U>X}|U|^{\sigma}\ f(U)dU$$

$$\int_{X>U}|X|^{S-\sigma-\rho-\alpha-\frac{m+1}{2}}|X-U|^{\rho+\alpha-\frac{m+1}{2}}e^{-tr(1-UX^{-1})}dX$$

On evaluating X -integral with the help of result given by Mathai

$$\int_0^1 e^{-tr(XZ)}\,|X|^{\delta-\frac{m+1}{2}}\,|I-X|^{\rho-\delta-\frac{m+1}{2}}\,dX$$

$$=\frac{\Gamma_m(\delta)\Gamma_m(\rho-\delta)}{\Gamma_m(\rho)}\,{}_1F_1[\delta;\rho;-Z] \qquad (3.5.7)$$

For $\mathrm{Re}(\delta)>\frac{m+1}{2},\mathrm{Re}(\rho)>\frac{m+1}{2},\mathrm{Re}(\rho-\delta)>\frac{m+1}{2}$

We obtain the required result.

Theorem5. If $f(X)\in L_P(0,\infty)\ 1\le P\le 2[or f(X)\in M_P(0,\infty)\ and\ P>2]$ where $\mathrm{Re}(\alpha)>\frac{m+1}{2},\mathrm{Re}(\delta)>-\frac{1}{P},\mathrm{Re}(\rho)>\frac{m+1}{2},\mathrm{Re}(\alpha-\rho)>\frac{m+1}{2},\frac{1}{P}+\frac{1}{Q}=1$ and $|\arg(I-a)|<\pi$ then

$$M\left\{K[^{\alpha}_{\delta,\rho,a};f(X)]\right\}=\frac{\Gamma_m\left(\delta+s+\frac{m+1}{2}\right)\Gamma_m(\rho+\alpha)}{\Gamma_m\left(\delta+s+\alpha+\rho+\frac{m+1}{2}\right)\Gamma_m(\rho)}$$

$${}_1F_1\left[\rho+\alpha;\delta+s+\alpha+\rho+\frac{m+1}{2};I\right]M[f(U)] \qquad (3.5.8)$$

Proof: Using the Mellin transform of

$$K[f(X)]=K\left[^{(a_r),(b_r)}_{\sigma,\rho,a};f(X)\right]=$$

$$\frac{|X|^{\delta}}{\Gamma_m(\rho)}\int_{U>X}|U|^{\delta}|X-U|^{\rho-\frac{m+1}{2}}\,G^{p,q}_{r,s}\left[a(I-XU^{-1})\Big|^{(a_r)}_{(b_r)}\right]f(U)dU \qquad (3.5.9)$$

We get $\quad M\left\{K[^{\alpha}_{\delta,\rho,a};f(X)]\right\}=$

$$\int_{X>0}\frac{|X|^{s-\frac{m+1}{2}}|X|^{\delta}}{\Gamma_m(\rho)}\left[\int_{U>X}|U|^{-\delta-\rho}|U-X|^{\rho-\frac{m+1}{2}}\,G^{1,0}_{0,1}\left[a(I-XU^{-1})\Big|^{-}_{(b_1)}\right]f(U)dU\right]dX \quad (3.5.10)$$

Changing the order of integration and evaluating X -integral with the help of (3.5.6), we obtain the required result.

When $M=1,N=0,P=0,Q=1,a=1$ in (3.5.6) and (3.5.9) reduces to the following from of operators:

$$R[f(X)]=R\left[^{\alpha,\beta}_{\sigma,\rho,1};f(X)\right]=$$

$$\frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)}\int_{0<U<X}|U|^{\sigma}|X-U|^{\rho-\frac{m+1}{2}}\,G^{1,0}_{0,1}\left[a(I-UX^{-1})\Big|^{\alpha}_{\beta}\right]f(U)dU \qquad (3.5.11)$$

And $K[f(X)] = K\left[{}^{\alpha,\beta}_{\sigma,\rho,1}; f(X)\right] =$

$$\frac{|X|^{-\sigma-\rho}}{\Gamma_m(\rho)} \int_{U>X} |U|^{-\delta-\sigma} |U-X|^{\rho-\frac{m+1}{2}} G^{1,0}_{0,1}\left[a(I-UX^{-1})\Big|^{\alpha}_{\beta}\right] f(U)dU \qquad (3.5.12)$$

Theorem6. If $f(X) \in L_P(0,\infty)$ $1 \le P \le 2[or f(X) \in M_P(0,\infty)\, and\, P > 2]$ where $\mathrm{Re}(\alpha) > \frac{m+1}{2}, \mathrm{Re}(\sigma) > -\frac{1}{Q}, \mathrm{Re}(\rho) > \frac{m+1}{2}, \mathrm{Re}(\alpha-\sigma) > \frac{m+1}{2}, \frac{1}{P}+\frac{1}{Q} = 1$ and $|\arg(I-a)| < \pi$ then

$$M\left\{R[{}^{\alpha}_{\sigma,\rho,1}; f(X)]\right\} = \frac{\Gamma_m(\sigma+\alpha-\beta-s)\Gamma_m(\rho+\beta)}{\Gamma_m(\sigma-s+\alpha+\rho)\Gamma_m(\rho)\Gamma_m(\alpha-\beta)} M[f(U)] \qquad (3.5.13)$$

Proof: Using the Mellin transform of (3.5.11), we get

$$M\left\{R[{}^{\alpha}_{\sigma,\rho,1}; f(X)]\right\} =$$

$$\int_{X>0} \frac{|X|^{-\sigma-\rho}|X|^{s-\frac{m+1}{2}}}{\Gamma_m(\rho)} \left[\int_{0<U<X} |U|^{\alpha}|X-U|^{\rho-\frac{m+1}{2}} G^{1,0}_{1,1}\left[a(I-UX^{-1})\Big|^{\alpha}_{\beta}\right] f(U)dU\right] dX$$

Changing the order of integration which is permissible under the conditions stated with the theorem, we obtain

$$M\left\{R[{}^{\alpha}_{\sigma,\rho,1}; f(X)]\right\} = \frac{1}{\Gamma_m(\rho)} \int_{U>0} |U|^{\sigma} f(U)dU$$

$$\int G^{1,0}_{1,1}\left[(I-XU^{-1})\Big|^{\alpha}_{\beta}\right] |X|^{s-\sigma-\rho-\frac{m+1}{2}} |X-U|^{\rho-\frac{m+1}{2}} dX$$

Using the result given by Mathai .

$$G^{1,0}_{1,1}\left[X\Big|^{\alpha}_{\beta}\right] = \frac{1}{\Gamma_m(\alpha-\beta)} |X|^{\beta} |I-U|^{\alpha-\beta-\frac{m+1}{2}} \qquad (3.5.14)$$

Provided $0 < X < I, \mathrm{Re}(\alpha-\beta) > \frac{m+1}{2}$

We get $M\left\{R[{}^{\alpha,\beta}_{\sigma,\rho,1}; f(X)]\right\} = \frac{1}{\Gamma_m(\rho)} \int_{U>0} |U|^{\sigma-\alpha-\beta-\frac{m+1}{2}} f(U)dU$

$$\frac{1}{\Gamma_m(\alpha-\beta)} \int_{X>U} |X|^{s-\sigma-\alpha} |X-U|^{\beta+\rho-\frac{m+1}{2}} dX$$

On evaluating X -integral with the help of the following result

$$\int_0^1 |X|^{\delta-\frac{m+1}{2}} |I-X|^{\rho-\frac{m+1}{2}} dX = \frac{\Gamma_m(\delta)\Gamma_m(\rho)}{\Gamma_m(\rho+\delta)} \qquad (3.5.15)$$

For $\operatorname{Re}(\delta) > 0, \operatorname{Re}(\rho) > \frac{m+1}{2}$

We arrive at the required result.

CHAPTER 4

ON A GENERALIZED PROBABILTY DISTRIBUTION,BOUNDARY VALUE PROBLEM AND EXPANSION FORMULA IN ASSOCIATION WITH A CERTAIN GENERALIZED HYPERGEOMETRIC FUNCTION WITH APPLICATION

In the present paper, a probability function $P(x)$ has been introduced in terms of the $\overline{H}$ -function and its properties are studied. It is shown that the classical non-central distributions such as, non-central chi-square, non-central Student- t, non-central F and almost all classical central continuous distributions can be obtained as special cases of this general density function. This general density function $P(x)$ is introduced with the hope that any density function, which can be represented in terms of any known special function as well as the density of the ratio of any two independent stochastic variables whose density functions can be represented in terms of any known special functions, is contained in $P(x)$ as a special case. The properties of $P(x)$, discussed in this paper, include the characteristic function, moments, recurrence relationship among moments and the distribution function.

Introduction

The $\overline{H}$ -function occurring in the paper will be defined and represented by Inayat-Hussain as follows:

$$\overline{H}^{M,N}_{P,Q}[z]=\overline{H}^{M,N}_{P,Q}\left[z\left|\begin{matrix}(a_j;\alpha_j;A_j)_{1,N},(a_j;\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\overline{\phi}(\xi)z^{\xi}d\xi \qquad (4.1.1)$$

where $$\overline{\phi}(\xi)=\frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\left\{\Gamma(1-a_j+\alpha_j\xi)\right\}^{A_j}}{\prod_{j=M+1}^{Q}\left\{\Gamma(1-b_j+\beta_j\xi)\right\}^{B_j}\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j\xi)} \qquad (4.1.2)$$

Which contains fractional powers of the gamma functions. Here, and throughout the paper $a_j(j=1,\ldots,p)$ and $b_j(j=1,\ldots,Q)$ are complex parameters, $\alpha_j\geq 0(j=1,\ldots,P),\beta_j\geq 0(j=1,\ldots,Q)$ (not all zero simultaneously) and exponents $A_j(j=1,\ldots,N)$ and $B_j(j=N+1,\ldots,Q)$ can take on non integer values.

The following sufficient condition for the absolute convergence of the defining integral for the $\overline{H}$ -function given by equation (4.1.1) have been given by (Buschman and Srivastava).

$$\Omega\equiv\sum_{j=1}^{M}\left|\beta_j\right|+\sum_{j=1}^{N}\left|A_j\alpha_j\right|-\sum_{j=M+1}^{Q}\left|\beta_jB_j\right|-\sum_{j=N+1}^{P}\left|\alpha_j\right|>0 \qquad (4.1.3)$$

and $$\left|\arg(z)\right|<\frac{1}{2}\pi\Omega \qquad (4.1.4)$$

The behavior of the $\overline{H}$ -function for small values of $|z|$ follows easily from a result recently given by (Rathie ,p.306,eq.(6.9)).

We have

$$\overline{H}_{P,Q}^{M,N}[z]=0\left(|z|^{\gamma}\right),\gamma=\min_{1\le j\le N}\left[\operatorname{Re}\left(b_j/\beta_j\right)\right],|z|\to 0 \tag{4.1.5}$$

If we take $A_j=1(j=1,2,...,N),B_j=1(j=M+1,...,Q)$ in (1.1), the function $\overline{H}_{P,Q}^{M,N}[.]$ reduces to the Fox's H -function .

The following series representation for the $\overline{H}$ -function will be required in the sequel (see Rathie, pp.305-306,eq.(6.8)):

$$\overline{H}_{P,Q}^{M,N}\left[z\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]=$$

$$\frac{\sum_{h=1}^{M}\sum_{r=0}^{\infty}\prod_{\substack{j=1\\j\ne h}}^{M}\Gamma\left(b_j-\beta_j\xi_{h,r}\right)\prod_{j=1}^{N}\left\{\Gamma\left(1-a_j+\alpha_j\xi_{h,r}\right)\right\}^{A_j}(-1)^r z^{\xi_{h,r}}}{\prod_{j=M+1}^{Q}\left\{\Gamma\left(1-b_j+\beta_j\xi_{h,r}\right)\right\}^{B_j}\prod_{j=N+1}^{P}\Gamma\left(a_j-\alpha_j\xi_{h,r}\right)r!\beta_h} \tag{4.1.6}$$

Where $\quad \xi_{h,r}=\dfrac{(b_h+r)}{\beta_h}$.

Some Definitions and Preliminary Results

Result 1.

$$\int_0^{\infty}\frac{x^{\lambda-1}}{(1+bx^k)^{\mu-1}}\overline{H}_{p,q}^{m,n}\left[\left(\frac{ax^k}{(1+bx^k)}\right)^s\left|\begin{matrix}A^*\\B^*\end{matrix}\right.\right]dx=$$

$$k^{-1}b^{-\frac{\lambda}{k}}\Gamma\left(\mu-1-\frac{\lambda}{k}\right)\overline{H}_{p+1,q+1}^{m,n+1}\left[\left(\frac{a}{b}\right)^s\left|\begin{matrix}\left(1-\frac{\lambda}{k},-s;1\right),A^*\\B^*,(2-\mu,-s;1)\end{matrix}\right.\right] \tag{4.2.1}$$

Where $\operatorname{Re}\left(\lambda-ks\dfrac{b_j}{\beta_j}\right)>0$ for $j=1,2,...,m;\operatorname{Re}(\lambda-k\mu+k)<0,k>0,b>0$ and $\operatorname{Re}(.)$ means the real part of $(.)$.

This result follows easily from the fact that,

$$\int_0^{\infty}x^{\lambda-1}(1+bx^k)^{\mu-1}dx=\frac{k^{-1}b^{-\frac{\lambda}{k}}\Gamma\left(\frac{\lambda}{k}\right)\Gamma\left(1-\mu-\frac{\lambda}{k}\right)}{\Gamma(1-\mu)} \tag{4.2.2}$$

Where $b,k>0,0<\operatorname{Re}\left(\dfrac{\lambda}{k}\right),\operatorname{Re}(1-\mu)$ on employing (4.1.1).

Result 2

$$\int_0^\infty e^{-dx}\frac{x^{\lambda-1}}{(1+bx^k)^{\mu-1}}\overline{H}_{p,q}^{m,n}\left[\left(\frac{ax^k}{(1+bx^k)}\right)^s\Big|_{B^*}^{A^*}\right]dx =$$

$$\sum_{r=0}^{\infty}\frac{(-1)^r d^r}{r!}k^{-1}b^{-\frac{(\lambda+r)}{k}}\Gamma\left(\mu-1-\frac{\lambda+r}{k}\right)\overline{H}_{p+1,q+1}^{m,n+1}\left[\left(\frac{a}{b}\right)^s\Big|_{B^*,(2-\mu,-s;1)}^{\left(1-\frac{\lambda+r}{k},-s;1\right),A^*}\right] \qquad (4.2.3)$$

Where $\mathrm{Re}(d)>0, \mathrm{Re}\left(\frac{\lambda}{k}\right)<\mathrm{Re}(\mu-1), \mathrm{Re}\left(\lambda-ks\frac{b_j}{\beta_j}\right)>0, b>0$.

The result follows by expanding e^{-dx} and integrating term by term by applying result 1.

Result 3

$$\int_0^\infty x^{\lambda-1}\left(1-\frac{x}{y}\right)^{\mu-1}\overline{H}_{p,q}^{m,n}\left[ax^v\Big|_{B^*}^{A^*}\right]dx = y^{\lambda}\Gamma(\mu)\overline{H}_{p+1,q+1}^{m,n+1}\left[\left(ay^v\right)\Big|_{B^*,(1-\lambda-\mu,-v;1)}^{(1-\lambda,-v;1),A^*}\right] \qquad (4.2.4)$$

Where $\mathrm{Re}\left(\lambda-v\frac{b_j}{\beta_j}\right)>0$ for $j=1,2,\ldots,m; \mathrm{Re}(\mu)>0$.

A General Probability Function

Here, we introduce a general probability density function $P(x)$ by using the most generalized function, namely the $\overline{H}$ -function. Such a generalized form is not necessary to obtain all the classical central and non-central distributions as special cases form this general distribution. Special cases which can be expressed in more compact form are given later. Without any loss of generality the function $P(x)$ is assumed to be non-negative since the parameters can always be chosen in such a way that $P(x)$ is always non-negative and still several parameters will be left to our choice so that several classes of non-negative functions can be obtained as special cases and the general nature of $P(x)$ is not lost either.

$$P(x)=\frac{e^{-dx}\frac{x^{\lambda-1}}{(1+bx^k)^{\mu-1}}\overline{H}_{p,q}^{m,n}\left[\left(\frac{ax^k}{(1+bx^k)}\right)^s\Big|_{B^*}^{A^*}\right]}{C(d)} \qquad (4.3.1)$$

For $x>0$ and $P(x)=0$ elsewhere, where

$$C(d)=\sum_{r=0}^{\infty}\frac{(-1)^r d^r}{r!}k^{-1}b^{-\frac{(\lambda+r)}{k}}\Gamma\left(\mu-1-\frac{\lambda+r}{k}\right)\overline{H}_{p+1,q+1}^{m,n+1}\left[\left(\frac{a}{b}\right)^s\Big|_{B^*,(2-\mu,-s;1)}^{\left(1-\frac{\lambda+r}{k},-s;1\right),A^*}\right] \qquad (4.3.2)$$

It should be pointed out the factor $x^{\mu-1}(1+bx^k)^{\mu-1}$ can be absorbed inside the $\overline{H}$ -function but it is written outside for convenience of manipulation later and when $d=0, C(d)$ can be written in a simple compact form as,

$$C(0)=k^{-1}b^{-\frac{\lambda}{k}}\Gamma\left(\mu-1-\frac{\lambda}{k}\right)\overline{H}^{m,n+1}_{p+1,q+1}\left[\left(\frac{a}{b}\right)^s\middle|^{\left(1-\frac{\lambda}{k},-s;1\right),A^*}_{B^*,(2-\mu,-s;1)}\right] \tag{4.3.3}$$

Then the probability function $P(x)$ reduces to

$$q(x)=\frac{\dfrac{x^{\lambda-1}}{(1+bx^k)^{\mu-1}}\overline{H}^{m,n}_{p,q}\left[\left(\dfrac{ax^k}{(1+bx^k)}\right)^s\middle|^{A^*}_{B^*}\right]}{C(0)} \tag{4.3.4}$$

Almost all the classical central and non-central distributions can be obtained from $q(x)$ which will be seen later. In order to obtain all the useful classical central and non-central distributions as special cases it is not necessary to take general density function in the form of $P(x)$. In the light of the result 2 and 3 of section 2 it is easily seen that

$$\int_0^\infty P(x)dx=1$$

Special Cases

If we put $A_j=B_j=1$ in (4.2.1), (4.2.3) and (4.2.4), we get the result given by Mathai and Saxena with a little simplification as:

$$\int_0^\infty \frac{x^{\lambda-1}}{(1+bx^k)^{\mu-1}}H^{m,n}_{p,q}\left[\left(\frac{ax^k}{(1+bx^k)}\right)^s\middle|^{(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q}}\right]dx=$$

$$k^{-1}b^{-\frac{\lambda}{k}}\Gamma\left(\mu-1-\frac{\lambda}{k}\right)H^{m,n+1}_{p+1,q+1}\left[\left(\frac{a}{b}\right)^s\middle|^{\left(1-\frac{\lambda}{k},-s\right),(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q},(2-\mu,-s)}\right] \tag{4.4.1}$$

With the conditions given in (2.1).

$$\int_0^\infty e^{-dx}\frac{x^{\lambda-1}}{(1+bx^k)^{\mu-1}}H^{m,n}_{p,q}\left[\left(\frac{ax^k}{(1+bx^k)}\right)^s\middle|^{(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q}}\right]dx=$$

$$\sum_{r=0}^\infty\frac{(-1)^r d^r}{r!}k^{-1}b^{-\frac{(\lambda+r)}{k}}\Gamma\left(\mu-1-\frac{\lambda+r}{k}\right)H^{m,n+1}_{p+1,q+1}\left[\left(\frac{a}{b}\right)^s\middle|^{\left(1-\frac{\lambda+r}{k},-s\right),(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q},(2-\mu,-s)}\right] \tag{4.4.2}$$

With the conditions given in (4.2.3).

$$\int_0^\infty x^{\lambda-1}\left(1-\frac{x}{y}\right)^{\mu-1}H^{m,n}_{p,q}\left[ax^v\middle|^{(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q}}\right]dx=y^\lambda\Gamma(\mu)H^{m,n+1}_{p+1,q+1}\left[\left(ay^v\right)\middle|^{(1-\lambda,-v),(a_j,\alpha_j)_{1,p}}_{(b_j,\beta_j)_{1,q},(1-\lambda-\mu,-v)}\right] \tag{4.4.3}$$

With the conditions given in (4.2.4).

Non-central Chi-square Distribution: The density function for the non-central chi-square is given by

$$m(x) = \begin{cases} e^{-\frac{\mu^2}{2\sigma^2}} \sum_{r=0}^{\infty} \frac{1}{r!\Gamma\left(r+\frac{k}{2}\right)} \left(\frac{\mu^2}{2\sigma^2}\right)^r \left(\frac{1}{2}\right)^{\left(r+\frac{k}{2}\right)} e^{-\frac{1}{2}x} x^{r+k-1} & for\ 0<x<\infty \\ 0, & elsewhere \end{cases} \qquad (4.4.4)$$

Put $d = \frac{1}{2}, \mu = 1, \lambda = \frac{k}{2}, s = k = 1, b = 0, b_1 = 0, b_2 = 1 - \frac{k}{2}, \beta_1 = \beta_2 = 1, a = \frac{\mu^2}{4\sigma^2}, A_j = 1 = B_j$ in $P(x)$.Using the formula,

$$\Gamma\left(\frac{k}{2}\right) G_{0,2}^{1,0}\left(-\frac{\mu^2}{4\sigma^2}\middle|\left(0, 1-\frac{k}{2}\right)\right) = {}_0F_1\left(\frac{k}{2}; \frac{\mu^2 x}{4\sigma^2}\right) \qquad (4.4.5)$$

And letting $b \to 0$, $P(x)$ reduces to $m(x)$ after a little simplification. In order to obtain the non-central F, non-central Beta, Student-t and a number of classical central distributions we need consider only $q(x)$ or $P(x)$ when $d = 0$ and it may be noticed that $q(x)$ is in a compact form.

Non-central F Distribution : The density function for non-central F is given by

$$g(x) = \begin{cases} e^{-\frac{\lambda^2}{2}} \sum_{m=0}^{\infty} \left(\frac{\lambda^2}{2}\right)^r \left(\frac{1}{r!}\right) \frac{\Gamma\left(\frac{k+m}{2}+r\right)}{\Gamma\left(\frac{k}{2}+r\right)\Gamma\left(\frac{m}{2}\right)} \frac{x^{r+\frac{k}{2}-1}}{(1+x)^{r+\frac{k+m}{2}}} & for\ x>0 \\ 0; & elsewhere \end{cases} \qquad (4.4.6)$$

By putting

$d = 0, \mu = \frac{k+m}{2} + 1, \lambda = \frac{k}{2}, s = k = 1, b = 1, b_1 = 1, b_2 = 1 - \frac{k}{2}, \beta_1 = \beta_2 = 1, a_1 = 1 - \frac{k+m}{2},$

$A_j = 1 = B_j, \alpha_1 = \beta_1 = \beta_2 = 1, m = n = p = 1, q = 2$, the $\overline{H}$ -function reduces to the G -function of the desired form here. Then by using the general properties that,

$$G_{1,2}^{1,1}\left(x\middle|{}_{(0,1-c)}^{(1-a)}\right) = \frac{\Gamma(a)\,{}_1F_1(a;z;-x)}{\Gamma(c)} \text{ and } G_{0,1}^{1,0}\left(x\middle|0\right) = e^{-x} \qquad (4.4.7)$$

$P(x)$ Reduces to $g(x)$. By a simple change of variables we get the non-central Beta distribution, with the density function,

$$g_1(x) = \begin{cases} e^{-\frac{\lambda^2}{2}} \frac{\Gamma\left(\frac{m+k}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{k}{2}\right)} x^{\frac{k}{2}-1} (1-x)^{\frac{m}{2}-1} {}_1F_1\left(\frac{k+m}{2}; \frac{k}{2}; \frac{\lambda^2 x}{2}\right); & for\ 0<x<1 \\ 0; & elsewhere \end{cases} \qquad (4.4.8)$$

It may be noted that the conditional distribution of the multiple correlation coefficient under the condition of given values of the observations on the variables in a multivariable normal case, is a non-central Beta distribution.

Non-central Student-t Distribution: The density function for the non-central Student-t distribution is given by:

$$h(x)=\frac{\nu^{\frac{\nu}{2}}e^{-\frac{\delta^2}{2}}\sum_{r=0}^{\infty}\Gamma\left(\frac{\nu+1+r}{2}\right)\left(\frac{\delta^2}{r!}\right)\left(\frac{2x^2}{\nu+x^2}\right)^{\frac{r}{2}}}{\Gamma\left(\frac{\nu}{2}\right)\left(\nu+x^2\right)^{\frac{\nu+1}{2}}};-\infty<x<\infty \qquad (4.4.9)$$

Where δ is the non-centrality parameter and k is the degrees of freedom. For convenience we will take the distribution in the folded form, that is

$$\begin{aligned} h_1(x) &= 2h(x)\ for\ x>0 \\ &= 0,\ elsewhere \end{aligned} \qquad (4.4.10)$$

Put $d=0,\mu=\frac{\nu+3}{2}+1,\lambda=1,b=1,b_1=0,\beta_1=1,a_1=\frac{1-\nu}{2},$ $A_j=1=B_j,\alpha_1=\frac{1}{2},m=n=p=q=1,k=2,$

replace b by $\frac{1}{\nu}$ and a by $\frac{2\delta^2}{\nu}$. Then $P(x)$ reduces to $h_1(x)$.

The generalized hypergeometric function: The authors in introduced a general probability distribution from where the following distributions were obtained as special cases: the general hypergeometric distribution, the generalized gamma, gamma, generalized F, F', Student-t, Beta, Exponential, Cauchy, Weibull, Raleigh, Waiting time and logistic. The density function employed was,

$$f(x)=\begin{cases}\dfrac{ea^{\frac{c}{e}}\Gamma(\alpha)\Gamma(\beta)\Gamma\left(\frac{\gamma-c}{e}\right)x^{c-1}}{\Gamma\left(\frac{c}{e}\right)\Gamma\left(\frac{\alpha-c}{e}\right)\Gamma\left(\frac{\beta-c}{e}\right)\Gamma(\gamma)}\,{}_2F_1\left(\alpha,\beta;\gamma;-ax^e\right);\ for\ x,c>0,\frac{\alpha-c}{e}>0,\frac{\beta-c}{e}>0 \\ 0;elsewhere\end{cases} \qquad (4.4.11)$$

This can be obtained as a special case from $P(x)$ by making the following substitutions. Put $\lambda=c,d=0,b=0,s=1,a_1=1-\alpha,a_2=1-\beta,b_1=0,b_2=1-\gamma,k=c,\alpha_1=\alpha_2=\beta_1=\beta_2=1$

$A_j=1=B_j$ .Using the formula

$$H_{2,2}^{1,2}\left[x\middle|_{(0,1),(1-c,1)}^{(1-a,1),(1-b,1)}\right]=\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}\,{}_2F_1(a,b;c;-x) \qquad (4.4.12)$$

We get $f(x)$ from $P(x)$ after a little simplification.

The Ratio Distribution: The ratio distribution can be obtained as:

$$f_2(x)=\begin{cases} r\dfrac{x^{-\sigma}}{2}\theta(\sigma_1)\theta(\sigma_2)\overline{H}_{p+q,p+q}^{m+n,m+n}\left[x^r\middle|_{B^*,(1-a_p,\alpha_p;1)}^{(1-b_q,\beta_q;1),A^*}\right]\ for\ x>0 \\ 0,\ elsewhere\end{cases} \qquad (4.4.13)$$

Where

$$\theta(\sigma_j)=\frac{\prod_{k=m+1}^{q}\left\{\Gamma\left(1-b_k-\beta_k\frac{\sigma_j}{r}\right)\right\}^{B_j}\prod_{k=n+1}^{p}\Gamma\left(a_k+\alpha_k\frac{\sigma_j}{r}\right)}{\prod_{k=1}^{n}\left\{\Gamma\left(1-a_k-\alpha_k\frac{\sigma_j}{r}\right)\right\}^{A_j}\prod_{k=n+1}^{p}\Gamma\left(b_k+\beta_k\frac{\sigma_j}{r}\right)} \qquad (4.4.14)$$

From the structure of $f_2(x)$ itself it is evident that $f_2(x)$ can be obtained from $P(x)$ by making suitable changes in the parameters. Thus $P(x)$ also contains the density function of the ratio of two independent stochastic variables whose density functions can be expressed in terms of any known special function.

The Characteristic Function and Moments

Since the characteristic function is defined as

$$\underline{\theta}(t) = E\left(e^{itx}\right) = \int_0^\infty e^{itx} P(x)dx \qquad (4.5.1)$$

Where $i = \sqrt{(-1)}$, it can be easily obtained by replacing the parameter d by $d - it$ and hence $\underline{\theta}(t) = \dfrac{C(d-it)}{C(d)}$ (4.5.2)

Where $C(d)$ is given in (4.3.2). Hence the moments and cumulates can be evaluated without much difficulty.

Moments: The v^{th} moment about the origin, M_v is obtained by replacing λ by $\lambda + v$ in (4.3.1) and then taking the ratio of the normalizing factors in $P(x)$. That is

$$M_v = \frac{C(d,\lambda+v)}{C(d,\lambda)} \qquad (4.5.3)$$

Where

$$C(d,\lambda) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} d^r k^{-1} b^{-\frac{\lambda+r}{k}} \Gamma\left(-1+\mu-\frac{\lambda+r}{k}\right) \overline{H}^{m,n+1}_{p+1,q+1}\left[\left(\frac{a}{b}\right)^s \middle| \begin{matrix} \left(1-\frac{\lambda+r}{k},s;1\right),A^* \\ B^*,(2-\mu,s;1) \end{matrix}\right] \qquad (4.5.4)$$

And if $d = 0$, this reduces to $\dfrac{C(0,\lambda+r)}{C(0,\lambda)}$, where

$$C(0,\lambda) = k^{-1} b^{-\frac{\lambda}{k}} \Gamma\left(-1+\mu-\frac{\lambda}{k}\right) \overline{H}^{m,n+1}_{p+1,q+1}\left[\left(\frac{a}{b}\right)^s \middle| \begin{matrix} \left(1-\frac{\lambda}{k},s;1\right),A^* \\ B^*,(2-\mu,s;1) \end{matrix}\right] \qquad (4.5.5)$$

A Recurrence Relationship: A recurrence relationship among M_v, M_{v-1} and M_{v+1} can be obtained by using the recurrence relationships for the $\overline{H}$ -function.

$$M_{\mu,v} = \frac{1}{C(d)} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} d^r k^{-1} b^{-\frac{\lambda+r+v}{k}} \Gamma\left(-1+\mu-\frac{\lambda+r+v}{k}\right) \overline{H}^{m,n+1}_{p+1,q+1}\left[\left(\frac{a}{b}\right)^s \middle| \begin{matrix} \left(1-\frac{\lambda+r+v}{k},s;1\right),A^* \\ B^*,(2-\mu,s;1) \end{matrix}\right] \qquad (4.5.6)$$

Where $C(d)$ is given in (4.3.2) . On applying the recurrence formula for the $\overline{H}$ -function.

$$\left(1-a_1+b_q\right)\overline{H}^{M,N}_{P,Q}\left[x\middle| \begin{matrix} (a_j;\alpha_j;A_j)_{1,N},(a_j;\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q-1},(b_Q,\alpha_1;B_Q) \end{matrix}\right] =$$

$$\overline{H}^{M,N}_{P,Q}\left[x\middle| \begin{matrix} (a_1-1,\alpha_1;A_1),(a_j;\alpha_j;A_j)_{2,N},(a_j;\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q-1},(b_Q,\alpha_1;B_Q) \end{matrix}\right] - \overline{H}^{M,N}_{P,Q}\left[x\middle| \begin{matrix} (a_j;\alpha_j;A_j)_{1,N},(a_j;\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q-1},(b_Q+1,\alpha_1;B_Q) \end{matrix}\right] \qquad (4.5.7)$$

To M_v of (4.5.3), we see that M_v is equal to

$$\frac{1}{C(d)}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}d^r k^{-1} b^{-\frac{\lambda+r+v}{k}}\left(-2+\mu-\frac{\lambda+r+v}{k}\right)\Gamma\left(-2+\mu-\frac{\lambda+r+v}{k}\right)$$

$$\overline{H}^{m,n+1}_{p+1,q+1}\left[\left(\frac{a}{b}\right)^s \middle|^{\left(1-\frac{\lambda+r+v}{k},s;1\right),A^*}_{B^*,(2-\mu,s;1)}\right]$$

$$=\frac{1}{C(d)}\sum_{r=0}^{\infty}(-1)^r d^r k^{-1} b^{-\frac{\lambda+r+v}{k}}\Gamma\left(-2+\mu-\frac{\lambda+r+v}{k}\right)$$

$$\left\{\overline{H}^{m,n+1}_{p+1,q+1}\left[\left(\frac{a}{b}\right)^s \middle|^{\left(1-\frac{\lambda+r+v}{k},s;1\right),A^*}_{B^*,(3-\mu,s;1)}\right]-\overline{H}^{m,n+1}_{p+1,q+1}\left[\left(\frac{a}{b}\right)^s \middle|^{\left(-\frac{\lambda+r+v}{k},s;1\right),A^*}_{B^*,(2-\mu,s;1)}\right]\right\}$$

Hence, we obtain

$$M_{\mu,v}=M_{\mu-1,v}-bM_{\mu,v+1} \tag{4.5.8}$$

The Distribution Function

The distribution function or the emulative density function

$$F(y)=\int_0^y P(x)dx$$

Can be obtained foe some special forms of $P(x)$. By putting $s=1$ and taking the limit $d\to 0$ and $b\to 0, P(x)$ reduces to the form

$$P_1(x)=\begin{cases} r\theta^{\frac{\lambda}{r}}\theta(\sigma)x^{\lambda-1}\overline{H}^{m,n}_{p,q}\left[ax^v \middle|^{A^*}_{B^*}\right] & for\ x>0 \\ 0, & elsewhere \end{cases} \tag{4.6.1}$$

By using result (2.2), we get

$$\int_0^y P_1(x)dx = r\theta^{\frac{\lambda}{r}}\theta(\lambda)y^{\lambda}\overline{H}^{m,n}_{p,q}\left[ay^v \middle|^{(1-\lambda,v;1),A^*}_{B^*,(-\lambda,v;1)}\right] \tag{4.6.2}$$

Where $\mathrm{Re}\left(\lambda+v\frac{b_j}{\beta_j}\right)>0;\ j=1,2,\ldots,m$ and $\theta(\lambda)$ is defined in (4.4.13). By using $F(y)$ we can obtain the distributions of order statistics and other related statistics which we will not discuss here.

In the present paper, a probability function $P(x)$ has been introduced in terms of the A-function and its properties are studied. It is shown that the classical non-central distributions such as, non-central chi-square, non-central Student- t, non-central F and almost all classical central continuous distributions can be obtained as special cases of this general density function. This general density function $P(x)$ is introduced with the hope that any density function, which can be represented in terms of any known special function as well as the density of the ratio of any two independent stochastic variables whose density functions can be represented in terms of any known special functions, is contained in $P(x)$ as a special case.

The properties of $P(x)$, discussed in this paper, include the characteristic function, moments, recurrence relationship among moments and the distribution function.

Introduction

${}_1(a_j,\alpha_j)_n$ Represents the set of n pairs of parameters the A-function was defined by Gautam as

$$A_{p,q}^{m,n}\left[x\left|\begin{matrix}{}_1(a_j,\alpha_j)_p\\ {}_1(b_j,\beta_j)_q\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_L f(s)x^s ds \qquad (4.7.1)$$

Where

$$f(s)=\frac{\prod_{j=1}^{m}\Gamma(a_j+\alpha_j s)\prod_{j=1}^{n}\Gamma(1-b_j-\beta_j s)}{\prod_{j=m+1}^{p}\Gamma(1-a_j-\alpha_j s)\prod_{j=n+1}^{q}\Gamma(b_j+\beta_j s)} \qquad (4.7.2)$$

The integral on the right hand side of (4.7.1) is convergent when $f>0$ and $|\arg(ux)|<\frac{f\pi}{2}$, where

$$f=\mathrm{Re}(\sum_{j=1}^{m}\alpha_j-\sum_{j=m+1}^{p}\alpha_j+\sum_{j=1}^{n}\beta_j-\sum_{j=n+1}^{q}\beta_j \;,\; u=\prod_{j=1}^{p}\alpha_j^{\alpha_j}\prod_{j=1}^{q}\beta_j^{-\beta_j} \qquad (4.7.3)$$

(4.7.1) reduces to H-function given by Fox the following relation

$$A_{p,q}^{n,m}\left[x\left|\begin{matrix}{}_1(1-a_j,\alpha_j)_p\\ {}_1(1-b_j,\beta_j)_q\end{matrix}\right.\right]=H_{p,q}^{m,n}\left[x\left|\begin{matrix}{}_1(a_j,\alpha_j)_p\\ {}_1(b_j,\beta_j)_q\end{matrix}\right.\right]$$

Some Definitions and Preliminary Results

Result 1.

$$\int_0^\infty \frac{x^{\lambda-1}}{(1+bx^k)^{\mu-1}}A_{p,q}^{m,n}\left[\left(\frac{ax^k}{(1+bx^k)}\right)^s\left|\begin{matrix}{}_1(a_j,\alpha_j)_p\\ {}_1(b_j,\beta_j)_q\end{matrix}\right.\right]dx=$$

$$k^{-1}b^{-\frac{\lambda}{k}}\Gamma\left(\mu-1-\frac{\lambda}{k}\right)A_{p,q}^{m,n}\left[\left(\frac{a}{b}\right)^s\left|\begin{matrix}\left(\frac{\lambda}{k},-s\right),{}_1(a_j,\alpha_j)_p\\ {}_1(b_j,\beta_j)_q,(-1+\mu,-s)\end{matrix}\right.\right] \qquad (4.8.1)$$

Where $\mathrm{Re}\left(\lambda-s\frac{1-b_j}{\beta_j}\right)>0$ for $j=1,2,\ldots,m;\mathrm{Re}(\lambda-k\mu+k)<0,k>0,b>0$ and $\mathrm{Re}(.)$ means the real part of $(.)$.

This result follows easily from the fact that,

$$\int_0^{\infty} x^{\lambda-1}(1+bx^k)^{\mu-1}dx = \frac{k^{-1}b^{-\frac{\lambda}{k}}\Gamma\left(\frac{\lambda}{k}\right)\Gamma\left(1-\mu-\frac{\lambda}{k}\right)}{\Gamma(1-\mu)} \tag{4.8.2}$$

Where $b,k>0, 0<\mathrm{Re}\left(\frac{\lambda}{k}\right), \mathrm{Re}(1-\mu)$ on employing (4.7.1).

Result 2

$$\int_0^{\infty} e^{-dx}\frac{x^{\lambda-1}}{(1+bx^k)^{\mu-1}} A_{p,q}^{m,n}\left[\left(\frac{ax^k}{(1+bx^k)}\right)^s \middle| \begin{matrix} {}_1(a_j,\alpha_j)_p \\ {}_1(b_j,\beta_j)_q \end{matrix}\right]dx =$$

$$\sum_{r=0}^{\infty}\frac{(-1)^r d^r}{r!}k^{-1}b^{-\frac{(\lambda+r)}{k}}\Gamma\left(\mu-1-\frac{\lambda+r}{k}\right) A_{p,q}^{m,n}\left[\left(\frac{a}{b}\right)^s \middle| \begin{matrix} \left(\frac{\lambda+r}{k},-s\right),{}_1(a_j,\alpha_j)_p \\ {}_1(b_j,\beta_j)_q,(-1+\mu,-s) \end{matrix}\right] \tag{4.8.3}$$

Where $\mathrm{Re}(d)>0, \mathrm{Re}\left(\frac{\lambda}{k}\right)<\mathrm{Re}(\mu-1), \mathrm{Re}\left(\lambda-s\frac{1-b_j}{\beta_j}\right)>0, b>0$. The result follows by expanding e^{-dx} and integrating term by term by applying result 1.

Result 3

$$\int_0^{\infty} x^{\lambda-1}\left(1-\frac{x}{y}\right)^{\mu-1} A_{p,q}^{m,n}\left[ax^v \middle| \begin{matrix} {}_1(a_j,\alpha_j)_p \\ {}_1(b_j,\beta_j)_q \end{matrix}\right]dx =$$

$$y^{\lambda}\Gamma(\mu)A_{p+1,q+1}^{m,n+1}\left[ay^v \middle| \begin{matrix} (\lambda,-v;1),{}_1(a_j,\alpha_j)_p \\ {}_1(b_j,\beta_j)_q,(\lambda+\mu,-v) \end{matrix}\right] \tag{4.8.4}$$

Where $\mathrm{Re}\left(\lambda-v\frac{1-b_j}{\beta_j}\right)>0$ for $j=1,2,\ldots,m; \mathrm{Re}(\mu)>0$.

A General Probability Function

Here, we introduce a general probability density function $P(x)$ by using the most generalized function, namely the A-function. Such a generalized form is not necessary to obtain all the classical central and non-central distributions as special cases form this general distribution. Special cases which can be expressed in more compact form are given later. Without any loss of generality the function $P(x)$ is assumed to be non-negative since the parameters can always be chosen in such a way that $P(x)$ is always non-negative and still several parameters will be left to our choice so that several classes of non-negative functions can be obtained as special cases and the general nature of $P(x)$ is not lost either.

$$P(x) = \frac{e^{-dx}\frac{x^{\lambda-1}}{(1+bx^k)^{\mu-1}} A_{p,q}^{m,n}\left[\left(\frac{ax^k}{(1+bx^k)}\right)^s \middle| \begin{matrix} {}_1(a_j,\alpha_j)_p \\ {}_1(b_j,\beta_j)_q \end{matrix}\right]}{C(d)} \tag{4.9.1}$$

For $x > 0$ and $P(x) = 0$ elsewhere, where

$$C(d) = \sum_{r=0}^{\infty} \frac{(-1)^r d^r}{r!} k^{-1} b^{-\frac{(\lambda+r)}{k}} \Gamma\left(\mu - 1 - \frac{\lambda+r}{k}\right) A^{m,n}_{p,q}\left[\left(\frac{a}{b}\right)^s \left| \begin{matrix} \left(\frac{\lambda+r}{k}, -s\right),_1 (a_j, \alpha_j)_p \\ _1(b_j, \beta_j)_q, (-1+\mu, -s) \end{matrix} \right.\right] \quad (4.9.2)$$

It should be pointed out the factor $x^{\mu-1}(1+bx^k)^{\mu-1}$ can be absorbed inside the A-function but it is written outside for convenience of manipulation later and when $d = 0, C(d)$ can be written in a simple compact form as,

$$C(0) = k^{-1} b^{-\frac{\lambda}{k}} \Gamma\left(\mu - 1 - \frac{\lambda}{k}\right) A^{m,n}_{p,q}\left[\left(\frac{a}{b}\right)^s \left| \begin{matrix} \left(\frac{\lambda}{k}, -s\right),_1 (a_j, \alpha_j)_p \\ _1(b_j, \beta_j)_q, (-1+\mu, -s) \end{matrix} \right.\right] \quad (4.9.3)$$

Then the probability function $P(x)$ reduces to

$$q(x) = \frac{\dfrac{x^{\lambda-1}}{(1+bx^k)^{\mu-1}} A^{m,n}_{p,q}\left[\left(\dfrac{ax^k}{(1+bx^k)}\right)^s \left| \begin{matrix} _1(a_j, \alpha_j)_p \\ _1(b_j, \beta_j)_q \end{matrix} \right.\right]}{C(0)} \quad (4.9.4)$$

Almost all the classical central and non-central distributions can be obtained from $q(x)$ which will be seen later. In order to obtain all the useful classical central and non-central distributions as special cases it is not necessary to take general density function in the form of $P(x)$. In the light of the result 2 and 3 of section 2 it is easily seen that

$$\int_0^{\infty} P(x) dx = 1$$

Special Cases

If we put $a_j = 1 - a_j, b_j = 1 - b_j$ in (4.8.1), (4.8.3) and (4.8.4), we get the result given by Mathai and Saxena with a little simplification as:

$$\int_0^{\infty} \frac{x^{\lambda-1}}{(1+bx^k)^{\mu-1}} H^{m,n}_{p,q}\left[\left(\frac{ax^k}{(1+bx^k)}\right)^s \left| \begin{matrix} (a_j,\alpha_j)_{1,p} \\ (b_j,\beta_j)_{1,q} \end{matrix} \right.\right] dx =$$

$$k^{-1} b^{-\frac{\lambda}{k}} \Gamma\left(\mu - 1 - \frac{\lambda}{k}\right) H^{m,n+1}_{p+1,q+1}\left[\left(\frac{a}{b}\right)^s \left| \begin{matrix} \left(1-\frac{\lambda}{k}, -s\right), (a_j,\alpha_j)_{1,p} \\ (b_j,\beta_j)_{1,q}, (2-\mu, -s) \end{matrix} \right.\right] \quad (4.10.1)$$

With the conditions given in (4.8.1).

$$\int_0^{\infty} e^{-dx} \frac{x^{\lambda-1}}{(1+bx^k)^{\mu-1}} H^{m,n}_{p,q}\left[\left(\frac{ax^k}{(1+bx^k)}\right)^s \left| \begin{matrix} (a_j,\alpha_j)_{1,p} \\ (b_j,\beta_j)_{1,q} \end{matrix} \right.\right] dx =$$

$$\sum_{r=0}^{\infty}\frac{(-1)^r d^r}{r!}k^{-1}b^{-\frac{(\lambda+r)}{k}}\Gamma\left(\mu-1-\frac{\lambda+r}{k}\right)H_{p+1,q+1}^{m,n+1}\left[\left(\frac{a}{b}\right)^s\left|\begin{matrix}\left(1-\frac{\lambda+r}{k},-s\right),(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q},(2-\mu,-s)\end{matrix}\right.\right] \qquad (4.10.2)$$

With the conditions given in (4.8.3).

$$\int_0^{\infty}x^{\lambda-1}\left(1-\frac{x}{y}\right)^{\mu-1}H_{p,q}^{m,n}\left[ax^v\left|\begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right.\right]dx=y^{\lambda}\Gamma(\mu)H_{p+1,q+1}^{m,n+1}\left[\left(ay^v\right)\left|\begin{matrix}(1-\lambda,-v),(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q},(1-\lambda-\mu,-v)\end{matrix}\right.\right]$$

(4.10.3)

With the conditions given in (4.8.4).

Non-central Chi-square Distribution: The density function for the non-central chi-square is given by

$$m(x)=\begin{cases}e^{-\frac{\mu^2}{2\sigma^2}}\sum_{r=0}^{\infty}\frac{1}{r!\Gamma\left(r+\frac{k}{2}\right)}\left(\frac{\mu^2}{2\sigma^2}\right)^r\left(\frac{1}{2}\right)^{\left(r+\frac{k}{2}\right)}e^{-\frac{1}{2}x}x^{r+k-1} & for\ 0<x<\infty\\ 0, & elsewhere\end{cases} \qquad (4.20.4)$$

Put $d=\frac{1}{2},\mu=1,\lambda=\frac{k}{2},s=k=1,b=0,b_1=0,b_2=1-\frac{k}{2},\beta_1=\beta_2=1,\ a=\frac{\mu^2}{4\sigma^2},a_j=1-a_j,b_j=1-b_j$ in $P(x)$.Using the formula,

$$\Gamma\left(\frac{k}{2}\right)G_{0,2}^{1,0}\left(-\frac{\mu^2}{4\sigma^2}\left|\left(0,1-\frac{k}{2}\right)\right.\right)={}_0F_1\left(\frac{k}{2};\frac{\mu^2x}{4\sigma^2}\right) \qquad (4.10.5)$$

And letting $b\to 0$, $P(x)$ reduces to $m(x)$ after a little simplification. In order to obtain the non-central F, non-central Beta, Student-t and a number of classical central distributions we need consider only $q(x)$ or $P(x)$ when $d=0$ and it may be noticed that $q(x)$ is in a compact form.

Non-central F Distribution : The density function for non-central F is given by

$$g(x)=\begin{cases}e^{-\frac{\lambda^2}{2}}\sum_{m=0}^{\infty}\left(\frac{\lambda^2}{2}\right)^r\left(\frac{1}{r!}\right)\frac{\Gamma\left(\frac{k+m}{2}+r\right)}{\Gamma\left(\frac{k}{2}+r\right)\Gamma\left(\frac{m}{2}\right)}\frac{x^{r+\frac{k}{2}-1}}{(1+x)^{r+\frac{k+m}{2}}} & for\ x>0\\ 0; & elsewhere\end{cases} \qquad (4.10.6)$$

By putting $d=0,\mu=\frac{k+m}{2}+1,\lambda=\frac{k}{2},s=k=1,b=1,b_1=1,b_2=1-\frac{k}{2},\beta_1=\beta_2=1,a_1=\frac{k+m}{2},$

$a_j=1-a_j,b_j=1-b_j,\alpha_1=\beta_1=\beta_2=1,m=n=p=1,q=2$, the A-function reduces to the G-function of the desired form here. Then by using the general properties that,

$$G_{1,2}^{1,1}\left(x\left|\begin{matrix}(1-a)\\(0,1-c)\end{matrix}\right.\right)=\frac{\Gamma(a)\,{}_1F_1(a;z;-x)}{\Gamma(c)}\ \text{and}\ G_{0,1}^{1,0}\left(x\left|0\right.\right)=e^{-x} \qquad (4.10.7)$$

$P(x)$ Reduces to $g(x)$. By a simple change of variables we get the non-central Beta distribution, with the density function,

$$g_1(x) = \begin{matrix} e^{-\frac{\lambda^2}{2}} \frac{\Gamma\left(\frac{m+k}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{k}{2}\right)} x^{\frac{k}{2}-1}(1-x)^{\frac{m}{2}-1} {}_1F_1\left(\frac{k+m}{2};\frac{k}{2};\frac{\lambda^2 x}{2}\right); \ for \ 0<x<1 \\ 0; \ elsewhere \end{matrix} \tag{4.10.8}$$

It may be noted that the conditional distribution of the multiple correlation coefficient under the condition of given values of the observations on the variables in a multivariable normal case, is a non-central Beta distribution.

Non-central Student-*t* Distribution: The density function for the non-central Student-t distribution is given by:

$$h(x) = \frac{\nu^{\frac{\nu}{2}} e^{-\frac{\delta^2}{2}} \sum_{r=0}^{\infty} \Gamma\left(\frac{\nu+1+r}{2}\right)\left(\frac{\delta^2}{r!}\right)\left(\frac{2x^2}{\nu+x^2}\right)^{\frac{r}{2}}}{\Gamma\left(\frac{\nu}{2}\right)\left(\nu+x^2\right)^{\frac{\nu+1}{2}}}; -\infty < x < \infty \tag{4.10.9}$$

Where δ is the non-centrality parameter and k is the degrees of freedom. For convenience we will take the distribution in the folded form, that is

$$\begin{aligned} h_1(x) &= 2h(x) \ for \ x > 0 \\ &= 0, \ elsewhere \end{aligned} \tag{4.10.10}$$

Put

$d = 0, \mu = \frac{\nu+3}{2} + 1, \lambda = 1, b = 1, b_1 = 0, \beta_1 = 1, a_1 = \frac{1-\nu}{2},$

$a_j = 1 - a_j, b_j = 1 - b_j, \alpha_1 = \frac{1}{2}, m = n = p = q = 1, k = 2$, replace b by $\frac{1}{\nu}$ and a by $\frac{2\delta^2}{\nu}$. Then $P(x)$ reduces to $h_1(x)$.

The generalized hypergeometric function: The authors in introduced a general probability distribution from where the following distributions were obtained as special cases: the general hypergeometric distribution, the generalized gamma, gamma, generalized F, F', Student-t, Beta, Exponential, Cauchy, Weibull, Raleigh, Waiting time and logistic. The density function employed was,

$$f(x) = \begin{matrix} \frac{e a^{\frac{c}{e}} \Gamma(\alpha)\Gamma(\beta)\Gamma\left(\frac{\gamma-c}{e}\right) x^{c-1}}{\Gamma\left(\frac{c}{e}\right)\Gamma\left(\frac{\alpha-c}{e}\right)\Gamma\left(\frac{\beta-c}{e}\right)\Gamma(\gamma)} {}_2F_1\left(\alpha,\beta;\gamma;-ax^e\right); \ for \ x, c>0, \frac{\alpha-c}{e}>0, \frac{\beta-c}{e}>0 \\ 0; elsewhere \end{matrix} \tag{4.10.11}$$

This can be obtained as a special case from $P(x)$ by making the following substitutions. Put
$\lambda = c, d = 0, b = 0, s = 1, a_1 = 1-\alpha, a_2 = 1-\beta, b_1 = 0, b_2 = 1-\gamma, k = c, \alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1$

$a_j = 1 - a_j, b_j = 1 - b_j$.Using the formula

$$H_{2,2}^{1,2}\left[x\Big|_{(0,1),(1-c,1)}^{(1-a,1),(1-b,1)}\right] = \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} {}_2F_1(a,b;c;-x) \tag{4.10.12}$$

We get $f(x)$ from $P(x)$ after a little simplification.

The Ratio Distribution: The ratio distribution can be obtained as:

$$f_2(x) = \begin{cases} r\dfrac{x^{-\sigma}}{2}\theta(\sigma_1)\theta(\sigma_2)\overline{H}_{p+q,p+q}^{m+n,m+n}\left[x^r \left| \begin{matrix} (1-b_q,\beta_q;1),A^* \\ B^*,(1-a_p,\alpha_p;1) \end{matrix} \right.\right] & for\ x>0 \\ 0, & elsewhere \end{cases} \qquad (4.10.13)$$

Where $$\theta(\sigma_j) = \frac{\prod_{k=m+1}^{q}\left\{\Gamma\left(1-b_k-\beta_k\frac{\sigma_j}{r}\right)\right\}^{B_j}\prod_{k=n+1}^{p}\Gamma\left(a_k+\alpha_k\frac{\sigma_j}{r}\right)}{\prod_{k=1}^{n}\left\{\Gamma\left(1-a_k-\alpha_k\frac{\sigma_j}{r}\right)\right\}^{A_j}\prod_{k=n+1}^{p}\Gamma\left(b_k+\beta_k\frac{\sigma_j}{r}\right)} \qquad (4.10.14)$$

From the structure of $f_2(x)$ itself it is evident that $f_2(x)$ can be obtained from $P(x)$ by making suitable changes in the parameters. Thus $P(x)$ also contains the density function of the ratio of two independent stochastic variables whose density functions can be expressed in terms of any known special function.

The Characteristic Function and Moments

Since the characteristic function is defined as

$$\underline{\theta}(t) = E\left(e^{itx}\right) = \int_0^\infty e^{itx}P(x)dx \qquad (4.11.1)$$

Where $i = \sqrt{(-1)}$, it can be easily obtained by replacing the parameter d by $d-it$ and

hence $\underline{\theta}(t) = \dfrac{C(d-it)}{C(d)}$ (4.11.2)

Where $C(d)$ is given in (3.2). Hence the moments and cumulates can be evaluated without much difficulty.

Moments: The v^{th} moment about the origin, M_v is obtained by replacing λ by $\lambda+v$ in (3.1) and then taking the ratio of the normalizing factors in $P(x)$. That is

$$M_v = \frac{C(d,\lambda+v)}{C(d,\lambda)} \qquad (4.11.3)$$

Where $$C(d,\lambda) = \sum_{r=0}^{\infty}\frac{(-1)^r}{r!}d^r k^{-1} b^{-\frac{\lambda+r}{k}}\Gamma\left(-1+\mu-\frac{\lambda+r}{k}\right)$$

$$A_{p,q}^{m,n}\left[\left(\frac{a}{b}\right)^s \left| \begin{matrix} \left(\frac{\lambda+r}{k},-s\right),_1(a_j,\alpha_j)_p \\ _1(b_j,\beta_j)_q,(-1+\mu,-s) \end{matrix} \right.\right] \qquad (4.11.4)$$

And if $d=0$, this reduces to $\dfrac{C(0,\lambda+r)}{C(0,\lambda)}$, where

$$C(0,\lambda) = k^{-1}b^{-\frac{\lambda}{k}}\Gamma\left(-1+\mu-\frac{\lambda}{k}\right) A_{p,q}^{m,n}\left[\left(\frac{a}{b}\right)^s \left| \begin{matrix} \left(\frac{\lambda}{k},-s\right),_1(a_j,\alpha_j)_p \\ _1(b_j,\beta_j)_q,(-1+\mu,-s) \end{matrix} \right.\right] \qquad (4.11.5)$$

A Recurrence Relationship: A recurrence relationship among M_v, M_{v-1} and M_{v+1} can be obtained by using the recurrence relationships for the A-function.

$$M_{\mu,v} = \frac{1}{C(d)} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} d^r k^{-1} b^{-\frac{\lambda+r+v}{k}} \Gamma\left(-1+\mu-\frac{\lambda+r+v}{k}\right)$$

$$A_{p+1,q+1}^{m,n+1}\left[\left(\frac{a}{b}\right)^s \middle| \begin{matrix} \left(\frac{\lambda}{k},-s\right),{}_1(a_j,\alpha_j)_p \\ {}_1(b_j,\beta_j)_q,(-1+\mu,-s) \end{matrix}\right] \qquad (4.11.6)$$

Where $C(d)$ is given in (4.9.2) . On applying the recurrence formula for the A-function.

$$\left(1-a_1+b_q\right) A_{p,q}^{m,n}\left[x \middle| \begin{matrix} {}_1(a_j,\alpha_j)_p \\ {}_1(b_j,\beta_j)_{q-1},(b_Q,\alpha_1) \end{matrix}\right] =$$

$$A_{P,Q}^{M,N}\left[x \middle| \begin{matrix} (a_1-1,\alpha_1;A_1),{}_1(a_j;\alpha_j)_{P-1} \\ {}_1(b_j,\beta_j)_Q,(b_Q,\alpha_1) \end{matrix}\right] - A_{P,Q}^{M,N}\left[x \middle| \begin{matrix} (a_j;\alpha_j;A_j)_{1,N},{}_1(a_j;\alpha_j)_{P-1} \\ {}_1(b_j,\beta_j)_Q,(b_Q+1,\alpha_1) \end{matrix}\right] \qquad (4.11.7)$$

To M_v of (5.3), we see that M_v is equal to

$$\frac{1}{C(d)} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} d^r k^{-1} b^{-\frac{\lambda+r+v}{k}} \left(-2+\mu-\frac{\lambda+r+v}{k}\right) \Gamma\left(-2+\mu-\frac{\lambda+r+v}{k}\right)$$

$$A_{p+1,q+1}^{m,n+1}\left[\left(\frac{a}{b}\right)^s \middle| \begin{matrix} \left(\frac{\lambda+r}{k},-s\right),{}_1(a_j,\alpha_j)_p \\ {}_1(b_j,\beta_j)_q,(-1+\mu,-s) \end{matrix}\right]$$

$$= \frac{1}{C(d)} \sum_{r=0}^{\infty} (-1)^r d^r k^{-1} b^{-\frac{\lambda+r+v}{k}} \Gamma\left(-2+\mu-\frac{\lambda+r+v}{k}\right)$$

$$\left\{ A_{p+1,q+1}^{m,n+1}\left[\left(\frac{a}{b}\right)^s \middle| \begin{matrix} \left(\frac{\lambda+r+v}{k},-s\right),{}_1(a_j,\alpha_j)_p \\ {}_1(b_j,\beta_j)_q,(-1+\mu,-s) \end{matrix}\right] - A_{p+1,q+1}^{m,n+1}\left[\left(\frac{a}{b}\right)^s \middle| \begin{matrix} \left(\frac{\lambda+r+v}{k},-s\right),{}_1(a_j,\alpha_j)_p \\ {}_1(b_j,\beta_j)_q,(-1+\mu,-s) \end{matrix}\right]\right\}$$

Hence, we obtain

$$M_{\mu,v} = M_{\mu-1,v} - bM_{\mu,v+1} \qquad (4.11.8)$$

The Distribution Function

The distribution function or the emulative density function

$$F(y) = \int_0^y P(x)dx$$

Can be obtained foe some special forms of $P(x)$. By putting $s=1$ and taking the limit $d \to 0$ and $b \to 0, P(x)$ reduces to the form

$$P_1(x) = \begin{cases} r\theta^{\frac{\lambda}{r}}\theta(\sigma)x^{\lambda-1}\overline{H}_{p,q}^{m,n}\left[ax^v \middle| \begin{smallmatrix} A^* \\ B^* \end{smallmatrix}\right] & for\ x>0 \\ 0, & elsewhere \end{cases} \tag{4.12.1}$$

By using result (2.2), we get

$$\int_0^y P_1(x)dx = r\theta^{\frac{\lambda}{r}}\theta(\lambda)y^{\lambda}A_{p,q}^{m,n}\left[ay^v \middle| \begin{matrix} (\lambda,v),_1 (a_j,\alpha_j)_p \\ _1(b_j,\beta_j)_q,(\lambda,v) \end{matrix}\right] \tag{4.12.2}$$

Where $\text{Re}\left(\lambda + v\dfrac{1-b_j}{\beta_j}\right) > 0;\ j=1,2,\ldots,m$ and $\theta(\lambda)$ is defined in (4.10.13). By using $F(y)$ we can obtain the distributions of order statistics and other related statistics which we will not discuss here.In this chapter, a probability function $P(x)$ has been introduced in terms of the I -function and its properties are studied. It has shown that the classical non-central distributions such as, non-central chi-square, non-central Student-t, non-central F and almost all classical central continuous distributions can be obtained as special cases of this general density function. This general density function $P(x)$ is introduced with the hope that any density function, which can be represented in terms of any known special function as well as the density of the ratio of any two independent stochastic variables whose density functions can be represented in terms of any known special functions, is contained in $P(x)$ as a special case. The various properties of $P(x)$, discussed in this paper, include the characteristic function, moments, recurrence relationship among moments and the distribution function.

Introduction

The I-function introduced by (Saxena (1982)) will be represented and defined as follows

$$I[Z] = I_{p_i,q_i:r}^{m,n}[Z] = I_{p_i,q_i:r}^{m,n}\left[z\middle| \begin{smallmatrix} (a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i} \\ (b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i} \end{smallmatrix}\right] = \frac{1}{2\pi\omega}\int_L \theta(s)ds \tag{4.13.1}$$

where $\omega = \sqrt{-1}$

$$\theta(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j - \beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}-\beta_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji},\alpha_{ji}s)\right\}} \tag{4.13.2}$$

$p_{i,}q_i(i=1,\ldots,r), m,n$ are integers satisfying $0 \le n \le p_i$, $0 \le m \le q_i,(i=1,\ldots,r), r$ is finite $\alpha_j,\beta_j,\alpha_{ij},\beta_{ji}$ are real and $a_{j,}b_j,a_{ji},b_{ji}$ are complex numbers such that

$\alpha_j(b_h+v) \ne \beta_h(a_j-v-k)$ for $v,k=01,2,\ldots$

We shall use the following notations:

$A^* = (a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}; B^* = (b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}$

Some Definitions and Preliminary Results

Result 1.

$$\int_0^\infty \frac{x^{\lambda-1}}{(1+bx^k)^{\mu-1}} I^{m,n}_{p_i,q_i:r}\left[\left(\frac{ax^k}{(1+bx^k)}\right)^s \Big|^{A*}_{B*}\right]dx =$$

$$k^{-1}b^{-\frac{\lambda}{k}}\Gamma\left(\mu-1-\frac{\lambda}{k}\right) I^{m,n+1}_{p_i+1,q_i+1:r}\left[\left(\frac{a}{b}\right)^s \Bigg|^{\left(1-\frac{\lambda}{k},-s\right),A*}_{B*,(2-\mu,-s)}\right] \qquad (4.14.1)$$

Where

$$\mathrm{Re}\left(\lambda - ks\frac{b_{ji}}{\beta_{ji}}\right) > 0; i = 1,2,\dots,r \text{ for } j = 1,2,\dots,m; \mathrm{Re}(\lambda - k\mu + k) < 0, k > 0, b > 0 \text{ and}$$

Re(.) means the real part of (.) .

Re(.) This result follows easily from the fact that,

$$\int_0^\infty x^{\lambda-1}(1+bx^k)^{\mu-1}dx = \frac{k^{-1}b^{-\frac{\lambda}{k}}\Gamma\left(\frac{\lambda}{k}\right)\Gamma\left(1-\mu-\frac{\lambda}{k}\right)}{\Gamma(1-\mu)} \qquad (4.14.2)$$

Where $b,k > 0, 0 < \mathrm{Re}\left(\frac{\lambda}{k}\right), \mathrm{Re}(1-\mu)$ on employing (4.13.1).

Result 2

$$\int_0^\infty e^{-dx}\frac{x^{\lambda-1}}{(1+bx^k)^{\mu-1}} I^{m,n}_{p_i,q_i:r}\left[\left(\frac{ax^k}{(1+bx^k)}\right)^s \Big|^{A*}_{B*}\right]dx =$$

$$\sum_{r=0}^\infty \frac{(-1)^r d^r}{r!}k^{-1}b^{-\frac{(\lambda+r)}{k}}\Gamma\left(\mu-1-\frac{\lambda+r}{k}\right)$$

$$I^{m,n+1}_{p_i+1,q_i+1:r}\left[\left(\frac{a}{b}\right)^s \Bigg|^{\left(1-\frac{\lambda+r}{k},-s\right),A*}_{B*,(2-\mu,-s)}\right] \qquad (4.14.3)$$

Where $\mathrm{Re}(d) > 0, \mathrm{Re}\left(\frac{\lambda}{k}\right) < \mathrm{Re}(\mu-1), \mathrm{Re}\left(\lambda - ks\frac{b_{ji}}{\beta_{ji}}\right) > 0, b > 0$.

The result follows by expanding e^{-dx} and integrating term by term by applying result 1.

Result 3

$$\int_0^\infty x^{\lambda-1}\left(1-\frac{x}{y}\right)^{\mu-1}\overline{H}^{m,n}_{p,q}\left[ax^v \Big|^{A*}_{B*}\right]dx =$$

$$y^{\lambda}\Gamma(\mu) I_{p_i+1,q_i+1:r}^{m,n+1}\left[\left(ay^{v}\right)\Big|_{B^*,(1-\lambda-\mu,-v)}^{(1-\lambda,-v),A^*}\right] \qquad (4.14.4)$$

Where $\operatorname{Re}\left(\lambda - v\dfrac{b_{ji}}{\beta_{ji}}\right) > 0; i = 1,2,\ldots,r$ for $j = 1,2,\ldots,m; \operatorname{Re}(\mu) > 0$.

A General Probability Function

Here, we introduce a general probability density function $P(x)$ by using the most generalized function, namely the I -function. Such a generalized form is not necessary to obtain all the classical central and non-central distributions as special cases form this general distribution. Special cases which can be expressed in more compact form are given later. Without any loss of generality the function $P(x)$ is assumed to be non-negative since the parameters can always be chosen in such a way that $P(x)$ is always non-negative and still several parameters will be left to our choice so that several classes of non-negative functions can be obtained as special cases and the general nature of $P(x)$ is not lost either.

$$P(x) = \frac{e^{-dx}\dfrac{x^{\lambda-1}}{(1+bx^k)^{\mu-1}} I_{p_i,q_i:r}^{m,n}\left[\left(\dfrac{ax^k}{(1+bx^k)}\right)^{s}\Big|_{B^*}^{A^*}\right]}{C(d)} \qquad (4.15.1)$$

For $x > 0$ and $P(x) = 0$ elsewhere, where

$$C(d) = \sum_{r=0}^{\infty}\frac{(-1)^r d^r}{r!} k^{-1} b^{-\frac{(\lambda+r)}{k}} \Gamma\left(\mu - 1 - \frac{\lambda+r}{k}\right)$$

$$I_{p_i+1,q_i+1:r}^{m,n+1}\left[\left(\frac{a}{b}\right)^{s}\Big|_{B^*,(2-\mu,-s)}^{\left(1-\frac{\lambda+r}{k},-s\right),A^*}\right] \qquad (4.15.2)$$

It should be pointed out the factor $x^{\mu-1}(1+bx^k)^{\mu-1}$ can be absorbed inside the I -function but it is written outside for convenience of manipulation later and when $d = 0, C(d)$ can be written in a simple compact form as,

$$C(0) = k^{-1} b^{-\frac{\lambda}{k}} \Gamma\left(\mu - 1 - \frac{\lambda}{k}\right) I_{p_i+1,q_i+1:r}^{m,n+1}\left[\left(\frac{a}{b}\right)^{s}\Big|_{B^*,(2-\mu,-s)}^{\left(1-\frac{\lambda}{k},-s\right),A^*}\right] \qquad (4.15.3)$$

Then the probability function $P(x)$ reduces to

$$q(x) = \frac{\dfrac{x^{\lambda-1}}{(1+bx^k)^{\mu-1}} I_{p_i,q_i:r}^{m,n}\left[\left(\dfrac{ax^k}{(1+bx^k)}\right)^{s}\Big|_{B^*}^{A^*}\right]}{C(0)} \qquad (4.15.4)$$

Almost all the classical central and non-central distributions can be obtained from which will be seen later. In order to obtain all the useful classical central and non-central distributions as special cases it is not necessary to take general density function in the form of $P(x)$. In the light of the result 2 and 3 of section 2 it is easily seen that

$$\int_0^\infty P(x)dx = 1$$

Special Cases

If we put $r = 1$ in (4.14.1), (4.14.3) and (4.14.4), we get the result given by Mathai and Saxena (1971) with a little simplification as:

$$\int_0^\infty \frac{x^{\lambda-1}}{(1+bx^k)^{\mu-1}} H_{p,q}^{m,n}\left[\left(\frac{ax^k}{(1+bx^k)}\right)^s \middle|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]dx =$$

$$k^{-1}b^{-\frac{\lambda}{k}}\Gamma\left(\mu-1-\frac{\lambda}{k}\right) H_{p+1,q+1}^{m,n+1}\left[\left(\frac{a}{b}\right)^s \middle|_{(b_j,\beta_j)_{1,q},(2-\mu,-s)}^{\left(1-\frac{\lambda}{k},-s\right),(a_j,\alpha_j)_{1,p}}\right] \qquad (4.16.1)$$

With the conditions given in (4.14.1).

$$\int_0^\infty e^{-dx} \frac{x^{\lambda-1}}{(1+bx^k)^{\mu-1}} H_{p,q}^{m,n}\left[\left(\frac{ax^k}{(1+bx^k)}\right)^s \middle|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]dx =$$

$$\sum_{r=0}^\infty \frac{(-1)^r d^r}{r!} k^{-1}b^{-\frac{(\lambda+r)}{k}}\Gamma\left(\mu-1-\frac{\lambda+r}{k}\right) H_{p+1,q+1}^{m,n+1}\left[\left(\frac{a}{b}\right)^s \middle|_{(b_j,\beta_j)_{1,q},(2-\mu,-s)}^{\left(1-\frac{\lambda+r}{k},-s\right),(a_j,\alpha_j)_{1,p}}\right] \qquad (4.16.2)$$

With the conditions given in (4.14.3).

$$\int_0^\infty x^{\lambda-1}\left(1-\frac{x}{y}\right)^{\mu-1} H_{p,q}^{m,n}\left[ax^v \middle|_{(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p}}\right]dx =$$

$$y^{\lambda}\Gamma(\mu) H_{p+1,q+1}^{m,n+1}\left[(ay^v)\middle|_{(b_j,\beta_j)_{1,q},(1-\lambda-\mu,-v)}^{(1-\lambda,-v),(a_j,\alpha_j)_{1,p}}\right] \qquad (4.16.3)$$

With the conditions given in (4.14.4).

Non-central Chi-square Distribution: The density function for the non-central chi-square is given by

$$m(x) = \begin{cases} e^{-\frac{\mu^2}{2\sigma^2}}\sum_{r=0}^\infty \frac{1}{r!\Gamma\left(r+\frac{k}{2}\right)}\left(\frac{\mu^2}{2\sigma^2}\right)^r\left(\frac{1}{2}\right)^{\left(r+\frac{k}{2}\right)} e^{-\frac{1}{2}x} x^{r+k-1} & for\ 0<x<\infty \\ 0, & elsewhere \end{cases} \qquad (4.16.4)$$

Put

$$d = \frac{1}{2}, \mu = 1, \lambda = \frac{k}{2}, s = k = 1, b = 0, b_1 = 0, b_2 = 1-\frac{k}{2}, \beta_1 = \beta_2 = 1, a = \frac{\mu^2}{4\sigma^2}, r = 1$$

in $P(x)$.Using the formula,

$$\Gamma\left(\frac{k}{2}\right)G_{0,2}^{1,0}\left(-\frac{\mu^2}{4\sigma^2}\middle|\left(0,1-\frac{k}{2}\right)\right) = {}_0F_1\left(\frac{k}{2};\frac{\mu^2 x}{4\sigma^2}\right) \qquad (4.16.5)$$

And letting $b \to 0, P(x)$ reduces to $m(x)$ after a little simplification. In order to obtain the non-central F , non-central Beta, Student-t and a number of classical central distributions we need consider only $q(x)$ or $P(x)$ when $d = 0$ and it may be noticed that

$q(x)$ is in a compact form.

Non-central F Distribution : The density function for non-central F is given by

$$g(x) = \begin{cases} e^{-\frac{\lambda^2}{2}} \sum_{m=0}^{\infty} \left(\frac{\lambda^2}{2}\right)^r \left(\frac{1}{r!}\right) \frac{\Gamma\left(\frac{k+m}{2}+r\right)}{\Gamma\left(\frac{k}{2}+r\right)\Gamma\left(\frac{m}{2}\right)} \frac{x^{r+\frac{k}{2}-1}}{(1+x)^{r+\frac{k+m}{2}}} & for\ x>0 \\ 0; & elsewhere \end{cases} \tag{4.16.6}$$

By putting $d = 0, \mu = \frac{k+m}{2} + 1, \lambda = \frac{k}{2}, s = k = 1, b = 1, b_1 = 1, b_2 = 1 - \frac{k}{2},$

$\beta_1 = \beta_2 = 1, a_1 = 1 - \frac{k+m}{2}, r = 1, \alpha_1 = \beta_1 = \beta_2 = 1, m = n = p = 1, q = 2$, the I -function

reduces to the G -function of the desired form here. Then by using the general properties that,

$$G_{1,2}^{1,1}\left(x \middle|_{(0,1-c)}^{(1-a)}\right) = \frac{\Gamma(a)\,{}_1F_1(a;z;-x)}{\Gamma(c)} \text{ and } G_{0,1}^{1,0}\left(x \middle| 0\right) = e^{-x} \tag{4.16.7}$$

Reduces to $g(x)$. By a simple change of variables we get the non-central Beta distribution, with the density function,

$$g_1(x) = \begin{cases} e^{-\frac{\lambda^2}{2}} \frac{\Gamma\left(\frac{m+k}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{k}{2}\right)} x^{\frac{k}{2}-1} (1-x)^{\frac{m}{2}-1} {}_1F_1\left(\frac{k+m}{2}; \frac{k}{2}; \frac{\lambda^2 x}{2}\right); & for\ 0<x<1 \\ 0; & elsewhere \end{cases} \tag{4.16.8}$$

It may be noted that the conditional distribution of the multiple correlation coefficient under the condition of given values of the observations on the variables in a multivariable normal case, is a non-central Beta distribution.

Non-central Student-t Distribution: The density function for the non-central Student-

t distribution is given by:

$$h(x) = \frac{\nu^{\frac{\nu}{2}} e^{-\frac{\delta^2}{2}} \sum_{r=0}^{\infty} \Gamma\left(\frac{\nu+1+r}{2}\right)\left(\frac{\delta^2}{r!}\right)\left(\frac{2x^2}{\nu+x^2}\right)^{\frac{r}{2}}}{\Gamma\left(\frac{\nu}{2}\right)\left(\nu+x^2\right)^{\frac{\nu+1}{2}}}; -\infty < x < \infty \tag{4.16.9}$$

Where δ is the non-centrality parameter and k is the degrees of freedom. For

convenience we will take the distribution in the folded form, that is

$$\begin{aligned} h_1(x) &= 2h(x)\ for\ \ x > 0 \\ &= 0,\ elsewhere \end{aligned} \tag{4.16.10}$$

Put $d=0, \mu=\frac{v+3}{2}+1, \lambda=1, b=1, b_1=0, \beta_1=1, a_1=\frac{1-v}{2}$, $r=1, \alpha_1=\frac{1}{2}, m=n=p=q=1, k=2$, replace b by $\frac{1}{v}$ and a by $\frac{2\delta^2}{v}$.

Then $P(x)$ reduces to $h_1(x)$.

The generalized hypergeometric function: The authors in (1966) introduced a general probability distribution from where the following distributions were obtained as special cases: the general hypergeometric distribution, the generalized gamma, gamma, generalized F, F', Student-t, Beta, Exponential, Cauchy, Weibull, Raleigh, Waiting time and logistic. The density function employed was,

$$f(x)=\begin{cases}\dfrac{ea^{\frac{c}{e}}\Gamma(\alpha)\Gamma(\beta)\Gamma\left(\frac{\gamma-c}{e}\right)x^{c-1}}{\Gamma\left(\frac{c}{e}\right)\Gamma\left(\frac{\alpha-c}{e}\right)\Gamma\left(\frac{\beta-c}{e}\right)\Gamma(\gamma)}\,{}_2F_1\left(\alpha,\beta;\gamma;-ax^e\right);\ for\ x,c>0,\frac{\alpha-c}{e}>0,\frac{\beta-c}{e}>0\\ 0;elsewhere\end{cases} \tag{4.16.11}$$

This can be obtained as a special case from $P(x)$ by making the following substitutions.

Put $\lambda=c, d=0, b=0, s=1, a_1=1-\alpha, a_2=1-\beta, b_1=0$,

$b_2=1-\gamma, k=c, \alpha_1=\alpha_2=\beta_1=\beta_2=1\ r=1$.

Using the formula

$$H_{2,2}^{1,2}\left[x\left|{}^{(1-a,1),(1-b,1)}_{(0,1),(1-c,1)}\right.\right]=\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}\,{}_2F_1(a,b;c;-x) \tag{4.16.12}$$

We get $f(x)$ from $P(x)$ after a little simplification.

The Ratio Distribution: The ratio distribution can be obtained as:

$$f_2(x)=\begin{cases} r\frac{x^{-\sigma}}{2}\theta(\sigma_1)\theta(\sigma_2)\overline{H}_{p+q,p+q}^{m+n,m+n}\left[x^r\left|{}^{(1-b_q,\beta_q;1),A^*}_{B^*,(1-a_p,\alpha_p;1)}\right.\right]\ for\ x>0\\ 0,\ elsewhere\end{cases} \tag{4.16.13}$$

Where

$$\theta(\sigma_j)=\frac{\prod_{k=m+1}^{q}\left\{\Gamma\left(1-b_k-\beta_k\frac{\sigma_j}{r}\right)\right\}^{B_j}\prod_{k=n+1}^{p}\Gamma\left(a_k+\alpha_k\frac{\sigma_j}{r}\right)}{\prod_{k=1}^{n}\left\{\Gamma\left(1-a_k-\alpha_k\frac{\sigma_j}{r}\right)\right\}^{A_j}\prod_{k=n+1}^{p}\Gamma\left(b_k+\beta_k\frac{\sigma_j}{r}\right)} \tag{4.16.14}$$

From the structure of $f_2(x)$ itself it is evident that $f_2(x)$ can be obtained from $P(x)$ by making suitable changes in the parameters. Thus $P(x)$ also contains the density function of the ratio of two independent stochastic variables whose density functions can be expressed in terms of any known special function.

The Characteristic Function and Moments

Since the characteristic function is defined as

$$\underline{\theta}(t) = E\left(e^{itx}\right) = \int_0^\infty e^{itx} P(x)dx \qquad (4.17.1)$$

Where $i = \sqrt{(-1)}$, it can be easily obtained by replacing the parameter d by $d - it$ and

Hence $\underline{\theta}(t) = \dfrac{C(d-it)}{C(d)}$ (4.17.2)

Where $C(d)$ is given in (4.15.2). Hence the moments and cumulates can be evaluated without much difficulty.

Moments: The v^{th} moment about the origin, M_v is obtained by replacing λ by $\lambda + v$ in (5.15.1) and then taking the ratio of the normalizing factors in $P(x)$. That is

$$M_v = \frac{C(d, \lambda + v)}{C(d, \lambda)} \qquad (4.17.3)$$

Where $C(d,\lambda) = \sum_{r=0}^{\infty} \dfrac{(-1)^r}{r!} d^r k^{-1} b^{-\frac{\lambda+r}{k}} \Gamma\left(-1+\mu-\dfrac{\lambda+r}{k}\right)$

$$I_{p_i+1,q_i+1:r}^{m,n+1}\left[\left(\frac{a}{b}\right)^s \middle| \begin{matrix} \left(1-\frac{\lambda+r}{k},s\right), A^* \\ B^*, (2-\mu, s) \end{matrix}\right] \qquad (4.17.4)$$

And if $d = 0$, this reduces to $\dfrac{C(0,\lambda+r)}{C(0,\lambda)}$, where

$$C(0,\lambda) = k^{-1} b^{-\frac{\lambda}{k}} \Gamma\left(-1+\mu-\frac{\lambda}{k}\right) \; I_{p_i+1,q_i+1:r}^{m,n+1}\left[\left(\frac{a}{b}\right)^s \middle| \begin{matrix} \left(1-\frac{\lambda}{k},s\right), A^* \\ B^*, (2-\mu, s) \end{matrix}\right] \qquad (4.17.5)$$

A Recurrence Relationship: A recurrence relationship among M_v, M_{v-1} and M_{v+1} can be obtained by using the recurrence relationships for the I -function.

$$M_{\mu,v} = \frac{1}{C(d)} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} d^r k^{-1} b^{-\frac{\lambda+r+v}{k}} \Gamma\left(-1+\mu-\frac{\lambda+r+v}{k}\right)$$

$$\overline{H}_{p+1,q+1}^{m,n+1}\left[\left(\frac{a}{b}\right)^s \middle| \begin{matrix} \left(1-\frac{\lambda+r+v}{k},s;1\right), A^* \\ B^*, (2-\mu, s;1) \end{matrix}\right] \qquad (4.17.6)$$

Where $C(d)$ is given in (4.15.2). On applying the recurrence formula for the I -function.

$$\left(1-a_1+b_q\right) I_{P_i,Q_i:r}^{M,N}\left[x \middle| \begin{matrix} (a_j;\alpha_j)_{1,N}, (a_{ji};\alpha_{ji})_{N+1,P} \\ (b_j,\beta_j)_{1,M}, (b_{ji},\beta_{ji})_{M+1,Q-1}, (b_Q,\alpha_1) \end{matrix}\right] =$$

$$I_{P_i,Q_i:r}^{M,N}\left[x \middle| \begin{matrix} (a_1-1,\alpha_1), (a_j;\alpha_j)_{2,N}, (a_{ji};\alpha_{ji})_{N+1,P} \\ (b_j,\beta_j)_{1,M}, (b_{ji},\beta_{ji})_{M+1,Q-1}, (b_Q,\alpha_1) \end{matrix}\right] - I_{P_i,Q_i:r}^{M,N}\left[x \middle| \begin{matrix} (a_j;\alpha_j)_{1,N}, (a_{ji};\alpha_{ji})_{N+1,P} \\ (b_j,\beta_j)_{1,M}, (b_{ji},\beta_{ji})_{M+1,Q-1}, (b_Q+1,\alpha_1) \end{matrix}\right] \qquad (4.17.7)$$

To M_v of (4.17.3), we see that M_v is equal to

$$\frac{1}{C(d)}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}d^r k^{-1} b^{-\frac{\lambda+r+v}{k}}\left(-2+\mu-\frac{\lambda+r+v}{k}\right)\Gamma\left(-2+\mu-\frac{\lambda+r+v}{k}\right)$$

$$I^{m,n+1}_{p_i+1,q_i+1:r}\left[\left(\frac{a}{b}\right)^s \middle| \begin{matrix}\left(1-\frac{\lambda+r+v}{k},s\right),A^* \\ B^*,(2-\mu,s)\end{matrix}\right]$$

$$=\frac{1}{C(d)}\sum_{r=0}^{\infty}(-1)^r d^r k^{-1} b^{-\frac{\lambda+r+v}{k}}\Gamma\left(-2+\mu-\frac{\lambda+r+v}{k}\right)$$

$$\left\{I^{m,n+1}_{p_i+1,q_i+1:r}\left[\left(\frac{a}{b}\right)^s \middle| \begin{matrix}\left(1-\frac{\lambda+r+v}{k},s\right),A^* \\ B^*,(3-\mu,s)\end{matrix}\right]-I^{m,n+1}_{p_i+1,q_i+1:r}\left[\left(\frac{a}{b}\right)^s \middle| \begin{matrix}\left(-\frac{\lambda+r+v}{k},s\right),A^* \\ B^*,(2-\mu,s)\end{matrix}\right]\right\}$$

Hence, we obtain

$$M_{\mu,v}=M_{\mu-1,v}-bM_{\mu,v+1} \tag{4.17.8}$$

The Distribution Function

The distribution function or the emulative density function

$$F(y)=\int_0^y P(x)dx$$

Can be obtained foe some special forms of $P(x)$. By putting $s=1$ and taking the limit

$d\to 0$ and $b\to 0, P(x)$ reduces to the form

$$P_1(x)=\begin{matrix} r\theta^{\frac{\lambda}{r}}\theta(\sigma)x^{\lambda-1}\overline{H}^{m,n}_{p,q}\left[ax^v\middle|\begin{matrix}A^*\\B^*\end{matrix}\right] \ for\ x>0 \\ 0,\ elsewhere\end{matrix} \tag{4.18.1}$$

By using result (5.14.2), we get

$$\int_0^y P_1(x)dx=r\theta^{\frac{\lambda}{r}}\theta(\lambda)y^{\lambda}I^{m,n}_{p_i,q_i:r}\left[ay^v\middle|\begin{matrix}(1-\lambda,v),A^*\\B^*,(-\lambda,v)\end{matrix}\right] \tag{4.18.2}$$

Where $\mathrm{Re}\left(\lambda+v\frac{b_{ji}}{\beta_{ji}}\right)>0; i=1,2,\dots,r;\ j=1,2,\dots,m$ and $\theta(\lambda)$ is defined in (4.16.13). By using $F(y)$ we can obtain the distributions of order statistics and other related statistics which we will not discuss here.

Introduction

The operational techniques are important tools to compute various problems in various fields of sciences which are used in the works of Chaurasia , Chandel, Agrawal and Kumar , Chandel and Sengar and Kumar to find out several results in various problems in different field of sciences and thus motivating by this work, we construct a model problem for temperature distribution in a rectangular plate under prescribed boundary conditions and then evaluate its solution involving A-function with product of general class of polynomials.

The general class of polynomials is defined by Srivastava and Panda as:

$$S_{n_1,\dots,n_r}^{m_1,\dots,m_r}(x_1,\dots,x_r)=\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_r=0}^{[n_r/m_r]}\frac{(-n_1)_{m_1k_1}}{k_1!}\cdots\frac{(-n_r)_{m_rk_r}}{k_r!}F[n_1,k_1;\dots;n_r,k_r]x_1^{k_1}\dots x_r^{k_r} \quad ,(4.19.1)$$

Where, $m_1,\dots,m_r$ are arbitrary positive integers and the coefficients $F[n_1,k_1;\dots;n_r,k_r]$ are arbitrary constants real or complex . Finally, we derive some new particular cases and find their applications also.

${}_1(a_j,\alpha_j)_n$ Represents the set of n pairs of parameters the A-function was defined by Gautam as

$$A_{p,q}^{m,n}\left[x\left|\begin{matrix}{}_1(a_j,\alpha_j)_p\\ {}_1(b_j,\beta_j)_q\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_L f(s)x^s ds \qquad (4.19.2)$$

Where

$$f(s)=\frac{\prod_{j=1}^{m}\Gamma(a_j+\alpha_j s)\prod_{j=1}^{n}\Gamma(1-b_j-\beta_j s)}{\prod_{j=m+1}^{p}\Gamma(1-a_j-\alpha_j s)\prod_{j=n+1}^{q}\Gamma(b_j+\beta_j s)} \qquad (4.19.3)$$

The integral on the right hand side of (4.19.2) is convergent when $f>0$ and $|\arg(ux)|<\frac{f\pi}{2}$, where

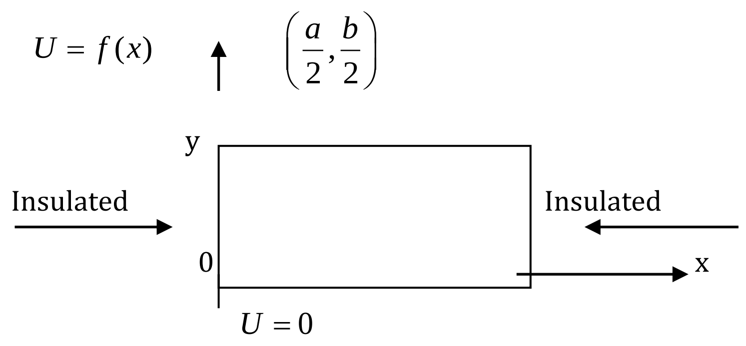
$$f=\mathrm{Re}(\sum_{j=1}^{m}\alpha_j-\sum_{j=m+1}^{p}\alpha_j+\sum_{j=1}^{n}\beta_j-\sum_{j=n+1}^{q}\beta_j \quad u=\prod_{j=1}^{p}\alpha_j^{\alpha_j}\prod_{j=1}^{q}\beta_j^{-\beta_j} \qquad (4.19.4)$$

(4.19.2) reduces to H-function given by Fox the following relation

$$A_{p,q}^{n,m}\left[x\left|\begin{matrix}{}_1(1-a_j,\alpha_j)_p\\ {}_1(1-b_j,\beta_j)_q\end{matrix}\right.\right]=H_{p,q}^{m,n}\left[x\left|\begin{matrix}{}_1(a_j,\alpha_j)_p\\ {}_1(b_j,\beta_j)_q\end{matrix}\right.\right]$$

A Boundary Value Problem

We consider a rectangular plate such that,



Where the boundary value conditions are:

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=a,\ 0<x<\frac{a}{2},0<y<\frac{b}{2} \tag{4.20.1}$$

$$\left.\frac{\partial U}{\partial x}\right|_{x=0}=\left.\frac{\partial U}{\partial x}\right|_{x=\frac{a}{2}}=0,0<y<\frac{b}{2} \tag{4.20.2}$$

$$U(x,0)=0,0<x<\frac{a}{2} \tag{4.20.3}$$

$$U\left(x,\frac{b}{2}\right)=f(x)=\left(\cos\frac{\pi x}{a}\right)S_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}\left[y_1\left(\cos\frac{\pi x}{a}\right)^{2\rho}\right] A_{p,q}^{m,n}\left[z\left(\cos\frac{\pi x}{a}\right)^{2\sigma}\left|\begin{matrix}{}_1(a_j,\alpha_j)_p\\ {}_1(b_j,\beta_j)_q\end{matrix}\right.\right] \tag{4.20.4}$$

Where, $0<x<\frac{a}{2}$ provided that $\mathrm{Re}(\eta)>-1,\sigma>0$. $U(x,y)$ is the temperature distribution in the rectangular plate at point (x,y).

Main Integral

In our investigations, we make an appeal to the modified formula due to Kumar as,

$$\int_0^{a/2}\left(\cos\frac{\pi x}{a}\right)^{\eta}\cos\frac{2m\pi x}{a}dx=\frac{a\Gamma(\eta+1)}{2^{\eta+1}\left(\frac{\eta}{2}+m+1\right)\left(\frac{\eta}{2}-m+1\right)} \tag{4.21.1}$$

Where, m is positive integer and $\mathrm{Re}(\eta)>-1$, then we evaluate an applicable integral

$$\int_0^{a/2}\left(\cos\frac{\pi x}{a}\right)^{\eta}\cos\frac{2m\pi x}{a}S_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}\left[y\left(\cos\frac{\pi x}{a}\right)^{2\rho}\right] A_{p,q}^{m,n}\left[z\left(\cos\frac{\pi x}{a}\right)^{2\sigma}\left|\begin{matrix}{}_1(a_j,\alpha_j)_p\\ {}_1(b_j,\beta_j)_q\end{matrix}\right.\right]dx$$

$$=\frac{a}{2^{\eta+1}}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_r=0}^{[n_r/m_r]}\frac{(-n)_{m_1k_1}}{k_1!}\cdots\frac{(-n)_{m_rk_r}}{k_r!} F[n_1,k_1;\ldots,n_r,k]A(k)\left(\frac{y}{4\rho}\right)^{k_1}\cdots\left(\frac{y}{4\rho}\right)^{k_r} \tag{4.21.2}$$

Where,

$$A(k) = A^{m,n+1}_{p+1,q+1}\left[\frac{z}{4^{\sigma}}\left|\begin{matrix}(1+\eta+2\rho k_1+...+2\rho k_r;2\sigma),_1(a_j,\alpha_j)_p\\ _1(b_j,\beta_j)_q,\left(1+\frac{\eta}{2}+m+\rho k_1+...+\rho k_r;\sigma\right)\end{matrix}\right.\right] \quad (4.21.3)$$

Provided that $F[n_1,k_1;...;n_r,k_r]$ are arbitrary functions of $n_1,k_1;...;n_r,k_r$, real or complex independent of x, y, ρ, the conditions of (4.20.4) and (4.21.1) are satisfied and

$$\mathrm{Re}\left(\eta+\sigma\frac{1-b_j}{\beta_j}\right) > -1, \ |\arg z| \le \frac{1}{2}\pi\Omega,$$

Solution of Boundary Value Problem

In this section, we obtain the solution of the boundary value problem stated in the section (2) as using (4.20.1), (4.20.2) and (4.20.3) with the help of the techniques referred to Zill as:

$$U(x,y) = A_0 y + \sum_{p=1}^{\infty} A_p \sinh\frac{2p\pi y}{a}\cos\frac{2p\pi x}{a}, 0<x<\frac{a}{2}, 0<y<\frac{a}{2} \quad (4.22.1)$$

For $y=\frac{b}{2}$, we find that

$$U\left(x,\frac{b}{2}\right) = f(x) = \frac{A_0 b}{2} + \sum_{p=1}^{\infty} A_p \sinh\frac{p\pi b}{a}\cos\frac{2p\pi x}{a}, 0<x<\frac{a}{2} \quad (4.22.2)$$

Now making an appeal to (2.22.4) and (4.22.2) and then interchanging both sides with respect to x from 0 to $\frac{a}{2}$, we derive,

$$A_0 = \frac{2}{b\sqrt{\pi}}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_r=0}^{[n_r/m_r]}(-n_1)_{m_1k_1}...(-n_r)_{m_rk_r}F[n_1,k_1;...;n_r,k_r]A_1(k)\frac{y^{k_1}}{k_1!}...\frac{y^{k_r}}{k_r!} \quad (4.22.3)$$

Where

$$A_1(k) = A^{m,n+1}_{p+1,q+1}\left[z\left|\begin{matrix}\left(\frac{1}{2}+\frac{\eta}{2}+\rho k_1+...+\rho k_r;\sigma\right),_1(a_j,\alpha_j)_p\\ _1(b_j,\beta_j)_q,\left(1+\frac{\eta}{2}+\rho k_1+...+\rho k_r;\sigma\right)\end{matrix}\right.\right] \quad (4.22.4)$$

Where all conditions of (4.20.4), (4.21.1) and (4.21.3) are satisfied. Again making an appeal to (2.22.4) and (4.22.2) and then multiplying by $\cos\frac{2m\pi x}{a}$ both sides and thus integrating that result with respect to x from 0 to $\frac{a}{2}$, we find,

$$A_m = \frac{1}{2^{\eta-1}\sinh\frac{p\pi b}{a}} \sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_r=0}^{[n_r/m_r]} \frac{(-n_1)_{m_1 k_1}}{k_1!} \cdots \frac{(-n_r)_{m_r k_r}}{k_r!} F[n_1, k_1; \ldots, n_r, k_r]$$

$$A(k)\left(\frac{y}{4^{\rho}}\right)^{k_1} \cdots \left(\frac{y}{4^{\rho}}\right)^{k_r} \tag{4.22.5}$$

Provided that all conditions of (4.20.4), (4.21.1) and (4.21.3) are satisfied. Finally, making an appeal to the result (4.22.1), (4.22.3) and (4.22.5), we derive the required solution of the boundary value problem,

$$U(x,y) = \frac{2y}{b\sqrt{\pi}} \sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_r=0}^{[n_r/m_r]} \left[\prod_{j=1}^{r}\left((-n_j)_{m_j k_j} \frac{y^{k_j}}{k_j!}\right)\right] F[n_1, k_1; \ldots; n_r, k_r] \; +$$

$$\sum_{m=1}^{\infty} \frac{\sinh\frac{2m\pi y}{a}\cos\frac{2m\pi x}{a}}{2^{\eta-1}\sinh\frac{m\pi b}{a}} \sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_r=0}^{[n_r/m_r]} \left[\prod_{j=1}^{r}\left((-n_j)_{m_j k_j}\left(\frac{y}{4^{\rho}}\right)^{k_j} \frac{1}{k_j!}\right)\right]$$

$$F[n_1, k_1; \ldots; n_r, k_r] A(k) \tag{4.22.6}$$

Provided that all conditions of (4.22.4), (4.21.1) and (4.21.3) are satisfied.

Expansion Formula

With the aid of (4.20.4) and (4.22.6) and then setting $y = \frac{b}{2}$, we evaluate the expansion formula

$$\left(\cos\frac{\pi x}{a}\right)^{\eta} S_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}\left[y\left(\cos\frac{\pi x}{a}\right)^{2\rho}\right] A_{p,q}^{m,n}\left[z\left(\cos\frac{\pi x}{a}\right)^{2\sigma}\middle|\begin{matrix} {}_1(a_j,\alpha_j)_p \\ {}_1(b_j,\beta_j)_q \end{matrix}\right]$$

$$= \frac{1}{\sqrt{\pi}} \sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_r=0}^{[n_r/m_r]} \left[\prod_{j=1}^{r}\left((-n_j)_{m_j k_j} \frac{y^{k_j}}{k_j!}\right)\right] F[n_1, k_1; \ldots; n_r, k_r] A(k)$$

$$\sum_{m=1}^{\infty} \frac{\cos\frac{2m\pi x}{a}}{2^{\eta-1}} \sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_r=0}^{[n_r/m_r]} \left[\prod_{j=1}^{r}\left((-n_j)_{m_j k_j}\left(\frac{y}{4^{\rho}}\right)^{k_j} \frac{1}{k_j!}\right)\right]$$

$$F[n_1, k_1; \ldots; n_r, k_r] A(k) \tag{4.23.1}$$

where $0 < x < \frac{a}{2}$, provided that all conditions of (4.20.4), (4.21.1) and (4.21.3) are satisfied.

Particular Cases and Applications

In this section, we do some setting of different parameters of our results and then drive some particular cases as stated here as taking $m_1 = \ldots = m_r = \gamma$ and

$$F[n_1,k_1;\dots;n_r,k_r]=\left(\frac{h}{(v)^{\gamma}}\right)^{k_1+\dots+k_r}\frac{1}{(1+p-n_1-\dots-n_r)_{\gamma(k_1+\dots+k_r)}}\text{ in (4.19.1)}$$

We get,

$$S_{n_1,\dots,n_r}^{\gamma,\dots,\gamma}[x_1,\dots,x_r]=\frac{(-v)^{-n_1\dots-n_r}}{(-p)_{n_1+\dots+n_r}}(x_1)^{n_1/\gamma}\dots(x_r)^{n_r/\gamma}H_{n_1,\dots,n_r}^{(h,\gamma,v,p)}[(x_1)^{-1/\gamma}\dots(x_r)^{-1/\gamma}] \quad (4.24.1)$$

And thus, we obtain an integral for product of a class of polynomials of several variables and cosine functions as

$$\int_0^{a/2}\left(\cos\frac{\pi x}{a}\right)^{\eta}\cos\frac{2m\pi x}{a}\frac{(-v)^{-n_1-\dots-n_r}}{(-p)_{n_1+\dots+n_r}}\left[y\left(\cos\frac{\pi x}{a}\right)^{2\rho}\right]^{n_1/\gamma}\dots\left[y\left(\cos\frac{\pi x}{a}\right)^{2\rho}\right]^{n_1/\gamma}$$

$$H_{n_1,\dots n_r}^{(h,\gamma,v,p)}\left[\left[y\left(\cos\frac{\pi x}{a}\right)^{2\rho}\right]^{r/\gamma}\right]A_{p,q}^{m,n}\left[z\left(\cos\frac{\pi x}{a}\right)^{2\sigma}\middle|\begin{matrix}{}_1(a_j,\alpha_j)_p\\ {}_1(b_j,\beta_j)_q\end{matrix}\right]dx$$

$$=\frac{a}{2^{\eta+1}}\sum_{k_1=0}^{[n_1/\gamma]}\dots\sum_{k_r=0}^{[n_r/\gamma]}\frac{(-n)_{\gamma k_1}}{k_1!}\dots\frac{(-n)_{\gamma k_r}}{k_r!}\left[\frac{h}{(-v)^{\gamma}}\right]^{k_1+\dots+k_r}$$

$$\frac{1}{(1+\rho-n_1-\dots-n_r)_{\gamma(k_1+\dots+k_r)}}A(k)\left(\frac{y}{4\rho}\right)^{k_1}\dots\left(\frac{y}{4\rho}\right)^{k_r} \qquad (4.24.2)$$

Provided that all conditions of (4.20.4), (4.21.1) and (4.21.2) are satisfied.

The solution of the given problem is

$$U(x,y)=\frac{2y}{b\sqrt{\pi}}\sum_{k_1=0}^{[n_1/\gamma]}\dots\sum_{k_r=0}^{[n_r/\gamma]}\left[\prod_{j=1}^{r}\left((-n_j)_{\gamma k_j}\left(\frac{hy}{(-v)^{\gamma}}\right)^{k_j}\frac{1}{k_j!}\right)\right]$$

$$\frac{1}{(1+p-n_1-\dots-n_r)_{\gamma(k_1+\dots+k_r)}}A_1(k)+\sum_{m=1}^{\infty}\frac{\sinh\frac{2m\pi y}{a}\cos\frac{2m\pi x}{a}}{2^{\eta-1}\sinh\frac{m\pi b}{a}}$$

$$\frac{1}{(1+p-n_1-\dots n_r)_{\gamma(k_1+\dots+k_r)}}A(k) \qquad (4.24.3)$$

When $0<x<\frac{a}{2},0<y<\frac{b}{2}$, provided that all conditions of (4.20.4), (4.21.1) and (4.21.3) are satisfied.

The expansion formula is

$$\left(\cos\frac{\pi x}{a}\right)^{\eta}\frac{(-v)^{-n_1-\dots-n_r}}{(-p)_{n_1+\dots+n_r}}\left[y\left(\cos\frac{\pi x}{a}\right)^{2\rho}\right]^{n_1/\gamma}\dots\left[y\left(\cos\frac{\pi x}{a}\right)^{2\rho}\right]^{n_1/\gamma}$$

$$H_{n_1,\dots n_r}^{(h,\gamma,v,p)}\left[\left[y\left(\cos\frac{\pi x}{a}\right)^{2\rho}\right]^{-r/\gamma}\right]A_{p,q}^{m,n}\left[z\left(\cos\frac{\pi x}{a}\right)^{2\sigma}\left|\begin{matrix}{}_1(a_j,\alpha_j)_p\\ {}_1(b_j,\beta_j)_q\end{matrix}\right.\right]$$

$$=\frac{1}{\sqrt{\pi}}\sum_{k_1=0}^{[n_1/\gamma]}\dots\sum_{k_r=0}^{[n_r/\gamma]}\left[\prod_{j=1}^{r}\left((-n_j)_{\gamma k_j}\left(\frac{hy}{(-v)^{\gamma}}\right)^{k_j}\frac{1}{k_j!}\right)\right]\frac{1}{(1+p-n_1-\dots-n_r)_{\gamma(k_1+\dots+k_r)}}$$

$$A_1(k)+\sum_{m=1}^{\infty}\frac{\cos\frac{2m\pi x}{a}}{2^{\eta-1}}\sum_{k_1=0}^{[n_1/\gamma]}\dots\sum_{k_r=0}^{[n_r/\gamma]}\left[\prod_{j=1}^{r}\left((-n_j)_{\gamma k_j}\left(\frac{hy}{(-v)^{\gamma}4^{\rho}}\right)^{k_j}\frac{1}{k_j!}\right)\right]$$

$$\frac{1}{(1+p-n_1-\dots n_r)_{\gamma(k_1+\dots+k_r)}}A(k) \qquad (4.24.4)$$

When $0<x<\frac{a}{2}$, provided that all conditions of (4.20.4), (4.21.1) and (4.21.3) are satisfied. Further, making an use of the result due to Chandel, Agarwal and Kumar (p. 27, wq. (1.4) and (1.5)).

$$\lim_{p\to\infty}H_{n_1,\dots,n_r}^{(h,\gamma,1,p)}\left(\frac{x_1}{p},\dots,\frac{x_r}{p}\right)=,\ \lim_{p\to\infty}H_{n_1,\dots,n_r}^{(h,\gamma,1/p,p)}(x_1,\dots,x_r)=g_{n_1}^{\gamma}(x_1,h)\dots g_{n_r}^{\gamma}(x_r,h) \qquad (4.24.5)$$

To the results (4.24.2), (4.24.3) and (4.24.4), we get another different relation in similar way.

Further again, applying the relation

$$g_n^2(x,-1/4)=2^{-n}H_n(x) \qquad (4.24.6)$$

To the above results, we evaluate another result for Hermite polynomials by same techniques.

Other special cases and applications of our results may be obtained by making use of the work of Chandel and Sengar , Srivastava and Karlsson and Srivastava and Manocha , due to lack of space we omit them.

Introduction

The operational techniques are important tools to compute various problems in various fields of sciences which are used in the works of Chaurasia , Chandel, Agrawal and Kumar , Chandel and Sengar and Kumar to find out several results in various problems in different field of sciences and thus motivating by this work, we construct a model problem for temperature distribution in a rectangular plate under prescribed boundary conditions and then evaluate its solution involving A-function with product of general class of polynomials.

The general class of polynomials is defined by Srivastava and Panda as:

$$S_{n_1,\dots,n_r}^{m_1,\dots,m_r}(x_1,\dots,x_r)=\sum_{k_1=0}^{[n_1/m_1]}\dots\sum_{k_r=0}^{[n_r/m_r]}\frac{(-n_1)_{m_1k_1}}{k_1!}\dots\frac{(-n_r)_{m_rk_r}}{k_r!}F[n_1,k_1;\dots;n_r,k_r]x_1^{k_1}\dots x_r^{k_r} \quad ,(4.25.1)$$

Where, $m_1,\dots,m_r$ are arbitrary positive integers and the coefficients $F[n_1,k_1;\dots;n_r,k_r]$ are arbitrary constants real or complex . Finally, we derive some new particular cases and find their applications also.

The I-function introduced by Saxena [1982] will be represented and defined as follows :

$$I[Z] = I^{m,n}_{p_i,q_i:r}[Z] = I^{m,n}_{p_i,q_i:r}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right] = \frac{1}{2\pi\omega}\int_L \chi(\xi)d\xi \qquad (4.25.2)$$

where $\omega = \sqrt{-1}$

$$\chi(\xi) = \frac{\prod_{j=1}^{m}\Gamma(b_j - \beta_j\xi)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\xi)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}-\beta_{ji})\prod_{j=n+1}^{p_i}\Gamma(a_{ji},\alpha_{ji})\right\}} \qquad (4.25.3)$$

$p_{i,}q_i(i=1,\ldots,r),m,n$ are integers satisfying $0 \le n \le p_i$, $0 \le m \le q_i,(i=1,\ldots,r),r$ is finite $\alpha_j,\beta_j,\alpha_{ij},\beta_{ji}$ are real and $a_{j,}b_j,a_{ji},b_{ji}$ are complex numbers such that

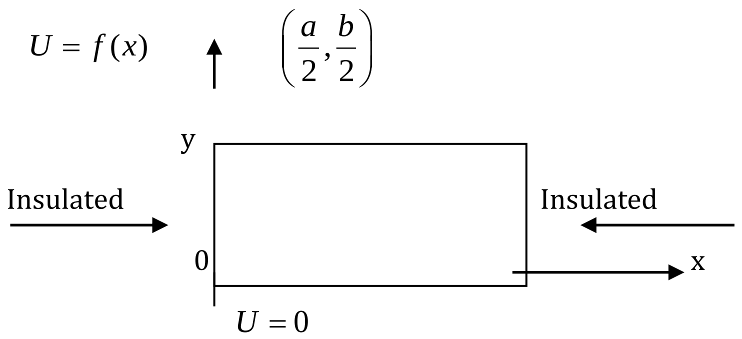
$\alpha_j(b_h+v) \ne \beta_h(a_j-v-k)$ for $v,k = 01,2,\ldots$

We shall use the following notations:

$A^* = (a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}; B^* = (b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}$

A Boundary Value Problem

We consider a rectangular plate such that,



Where the boundary value conditions are:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = a,\ 0 < x < \frac{a}{2}, 0 < y < \frac{b}{2} \qquad (4.26.1)$$

$$\left.\frac{\partial U}{\partial x}\right|_{x=0} = \left.\frac{\partial U}{\partial x}\right|_{x=\frac{a}{2}} = 0, 0 < y < \frac{b}{2} \qquad (4.26.2)$$

$$U(x,0) = 0, 0 < x < \frac{a}{2} \qquad (4.26.3)$$

$$U\left(x,\frac{b}{2}\right) = f(x) = \left(\cos\frac{\pi x}{a}\right)S^{m_1,\ldots,m_r}_{n_1,\ldots,n_r}\left[y_1\left(\cos\frac{\pi x}{a}\right)^{2\rho}\right]$$
$$A^{m,n}_{p,q}\left[z\left(\cos\frac{\pi x}{a}\right)^{2\sigma}\left|\begin{matrix}{}_1(a_j,\alpha_j)_p\\{}_1(b_j,\beta_j)_q\end{matrix}\right.\right] \qquad (4.26.4)$$

Where, $0 < x < \frac{a}{2}$ provided that $\mathrm{Re}(\eta) > -1, \sigma > 0$.

$U(x,y)$ is the temperature distribution in the rectangular plate at point (x,y).

Main Integral

In our investigations, we make an appeal to the modified formula due to Kumar as,

$$\int_0^{a/2} \left(\cos\frac{\pi x}{a}\right)^{\eta} \cos\frac{2m\pi x}{a}\,dx = \frac{a\Gamma(\eta+1)}{2^{\eta+1}\left(\frac{\eta}{2}+m+1\right)\left(\frac{\eta}{2}-m+1\right)} \qquad (4.27.1)$$

Where, m is positive integer and $\mathrm{Re}(\eta) > -1$, then we evaluate an applicable integral

$$\int_0^{a/2} \left(\cos\frac{\pi x}{a}\right)^{\eta} \cos\frac{2m\pi x}{a} S_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}\left[y\left(\cos\frac{\pi x}{a}\right)^{2\rho}\right]$$

$$I_{p_i,q_i:r}^{m,n}\left[z\left(\cos\frac{\pi x}{a}\right)^{2\sigma}\left|\begin{matrix}A^*\\B^*\end{matrix}\right.\right]dx$$

$$= \frac{a}{2^{\eta+1}} \sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_r=0}^{[n_r/m_r]} \frac{(-n)_{m_1k_1}}{k_1!} \cdots \frac{(-n)_{m_rk_r}}{k_r!} F[n_1,k_1;\ldots,n_r,k]I(k)\left(\frac{y}{4\rho}\right)^{k_1} \cdots \left(\frac{y}{4\rho}\right)^{k_r}$$

(4.27.2)

Where,

$$I(k) = I_{p_i+1,q_i+1:r}^{m,n+1}\left[\frac{z}{4^{\sigma}}\left|\begin{matrix}(-\eta-2\rho k_1-\ldots-2\rho k_r;2\sigma),A^*\\B^*,\left(-\frac{\eta}{2}-m-\rho k_1-\ldots-\rho k_r;\sigma\right)\end{matrix}\right.\right] \qquad (4.27.3)$$

Provided that $F[n_1,k_1;\ldots;n_r,k_r]$ are arbitrary functions of $n_1,k_1;\ldots;n_r,k_r$, real or complex independent of x,y,ρ, the conditions of (4.26.4) and (4.27.1) are satisfied and

$$\mathrm{Re}\left(\eta+\sigma\frac{b_{ji}}{\beta_{ji}}\right) > -1, \ |\arg z| \le \frac{1}{2}\pi\Omega,$$

Where $\Omega \equiv \sum_{j=1}^{n}\alpha_j + \sum_{j=1}^{m}\beta_j - \sum_{j=n+1}^{p_i}\alpha_{ji} - \sum_{j=m+1}^{q_i}\beta_{ji} > 0$

Solution of Boundary Value Problem

In this section, we obtain the solution of the boundary value problem stated in the section (2) as using (4.26.1), (4.26.2) and (4.26.3) with the help of the techniques referred to Zill as:

$$U(x,y) = A_0 y + \sum_{p=1}^{\infty} A_p \sinh\frac{2p\pi y}{a}\cos\frac{2p\pi x}{a}, 0 < x < \frac{a}{2}, 0 < y < \frac{a}{2} \qquad (4.28.1)$$

For $y=\frac{b}{2}$, we find that

$$U\left(x,\frac{b}{2}\right)=f(x)=\frac{A_0 b}{2}+\sum_{p=1}^{\infty}A_p \sinh\frac{p\pi b}{a}\cos\frac{2p\pi x}{a}, 0<x<\frac{a}{2} \qquad (4.28.2)$$

Now making an appeal to (4.26.4) and (4.28.2) and then interchanging both sides with respect to x from 0 to $\frac{a}{2}$, we derive,

$$A_0=\frac{2}{b\sqrt{\pi}}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_r=0}^{[n_r/m_r]}(-n_1)_{m_1k_1}\cdots(-n_r)_{m_rk_r}$$

$$F[n_1,k_1;\ldots;n_r,k_r]I_1(k)\frac{y^{k_1}}{k_1!}\cdots\frac{y^{k_r}}{k_r!} \qquad (4.28.3)$$

Where

$$I_1(k)=I^{m,n+1}_{p_i+1,q_i+1:r}\left[z\left|\begin{matrix}\left(-\frac{1}{2}-\frac{\eta}{2}-\rho k_1-\ldots-\rho k_r;\sigma\right),A^* \\ B^*,\left(-\frac{\eta}{2}-\rho k_1-\ldots-\rho k_r;\sigma\right)\end{matrix}\right.\right] \qquad (4.28.4)$$

Where all conditions of (2.26.4), (4.27.1) and (4.27.3) are satisfied.

Again making an appeal to (4.26.4) and (4.28.2) and then multiplying by $\cos\frac{2m\pi x}{a}$ both sides and thus integrating that result with respect to x from 0 to $\frac{a}{2}$, we find,

$$A_m=\frac{1}{2^{\eta-1}\sinh\frac{p\pi b}{a}}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_r=0}^{[n_r/m_r]}\frac{(-n_1)_{m_1k_1}}{k_1!}\cdots\frac{(-n_r)_{m_rk_r}}{k_r!}F[n_1,k_1;\ldots,n_r,k_r]$$

$$I(k)\left(\frac{y}{4^{\rho}}\right)^{k_1}\cdots\left(\frac{y}{4^{\rho}}\right)^{k_r} \qquad (4.28.5)$$

Provided that all conditions of (4.26.4), (4.27.1) and (4.27.3) are satisfied.

Finally, making an appeal to the result (4.28.1), (4.28.3) and (4.28.5), we derive the required solution of the boundary value problem,

$$U(x,y)=\frac{2y}{b\sqrt{\pi}}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_r=0}^{[n_r/m_r]}\left[\prod_{j=1}^{r}\left((-n_j)_{m_jk_j}\frac{y^{k_j}}{k_j!}\right)\right]F[n_1,k_1;\ldots;n_r,k_r]\ +$$

$$\sum_{m=1}^{\infty}\frac{\sinh\frac{2m\pi y}{a}\cos\frac{2m\pi x}{a}}{2^{\eta-1}\sinh\frac{m\pi b}{a}}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_r=0}^{[n_r/m_r]}\left[\prod_{j=1}^{r}\left((-n_j)_{m_jk_j}\left(\frac{y}{4^{\rho}}\right)^{k_j}\frac{1}{k_j!}\right)\right]$$

$$F[n_1,k_1;...;n_r,k_r]I(k) \qquad (4.28.6)$$

Provided that all conditions of (4.26.4), (4.27.1) and (4.27.3) are satisfied.

Expansion Formula

With the aid of (4.26.4) and (4.28.6) and then setting $y=\frac{b}{2}$, we evaluate the expansion formula

$$\left(\cos\frac{\pi x}{a}\right)^{\eta}S_{n_1,...,n_r}^{m_1,...,m_r}\left[y\left(\cos\frac{\pi x}{a}\right)^{2\rho}\right]I_{p_i,q_i:r}^{m,n}\left[z\left(\cos\frac{\pi x}{a}\right)^{2\sigma}\middle|\begin{matrix}A*\\B*\end{matrix}\right]$$

$$=\frac{1}{\sqrt{\pi}}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_r=0}^{[n_r/m_r]}\left[\prod_{j=1}^{r}\left((-n_j)_{m_jk_j}\frac{y^{k_j}}{k_j!}\right)\right]F[n_1,k_1;...;n_r,k_r]I(k)$$

$$\sum_{m=1}^{\infty}\frac{\cos\frac{2m\pi x}{a}}{2^{\eta-1}}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_r=0}^{[n_r/m_r]}\left[\prod_{j=1}^{r}\left((-n_j)_{m_jk_j}\left(\frac{y}{4^{\rho}}\right)^{k_j}\frac{1}{k_j!}\right)\right]$$

$$F[n_1,k_1;...;n_r,k_r]I(k) \qquad (4.29.1)$$

where $0<x<\frac{a}{2}$,provided that all conditions of (4.26.4), (4.27.1) and (4.27.3) are satisfied.

Particular Cases and Applications

In this section, we do some setting of different parameters of our results and then drive some particular cases as stated here as taking $m_1=...=m_r=\gamma$ and

$$F[n_1,k_1;...;n_r,k_r]=\left(\frac{h}{(v)^{\gamma}}\right)^{k_1+...+k_r}\frac{1}{(1+p-n_1-...-n_r)_{\gamma(k_1+...+k_r)}} \text{ in (4.25.1)}$$

We get,

$$S_{n_1,...,n_r}^{\gamma,...,\gamma}[x_1,...,x_r]=\frac{(-v)^{-n_1...-n_r}}{(-p)_{n_1+...+n_r}}(x_1)^{n_1/\gamma}...(x_r)^{n_r/\gamma}H_{n_1,...,n_r}^{(h,\gamma,v,p)}[(x_1)^{-1/\gamma}...(x_r)^{-1/\gamma}] \quad (4.30.1)$$

And thus, we obtain an integral for product of a class of polynomials of several variables and cosine functions as

$$\int_0^{a/2}\left(\cos\frac{\pi x}{a}\right)^{\eta}\cos\frac{2m\pi x}{a}\frac{(-v)^{-n_1-\ldots-n_r}}{(-p)_{n_1+\ldots,+n_r}}\left[y\left(\cos\frac{\pi x}{a}\right)^{2\rho}\right]^{n_1/\gamma}\ldots\left[y\left(\cos\frac{\pi x}{a}\right)^{2\rho}\right]^{n_1/\gamma}$$

$$H_{n_1,\ldots n_r}^{(h,\gamma,v,p)}\left[\left[y\left(\cos\frac{\pi x}{a}\right)^{2\rho}\right]^{r/\gamma}\right]I_{p_i,q_i:r}^{m,n}\left[z\left(\cos\frac{\pi x}{a}\right)^{2\sigma}\middle|\begin{matrix}A*\\B*\end{matrix}\right]dx$$

$$=\frac{a}{2^{\eta+1}}\sum_{k_1=0}^{[n_1/\gamma]}\ldots\sum_{k_r=0}^{[n_r/\gamma]}\frac{(-n)_{\gamma k_1}}{k_1!}\ldots\frac{(-n)_{\gamma k_r}}{k_r!}\left[\frac{h}{(-v)^{\gamma}}\right]^{k_1+\ldots+k_r}$$

$$\frac{1}{(1+\rho-n_1-\ldots-n_r)_{\gamma(k_1+\ldots+k_r)}}I(k)\left(\frac{y}{4\rho}\right)^{k_1}\ldots\left(\frac{y}{4\rho}\right)^{k_r} \qquad (4.30.2)$$

Provided that all conditions of (4.26.4), (4.27.1) and (4.27.2) are satisfied.

The solution of the given problem is

$$U(x,y)=\frac{2y}{b\sqrt{\pi}}\sum_{k_1=0}^{[n_1/\gamma]}\ldots\sum_{k_r=0}^{[n_r/\gamma]}\left[\prod_{j=1}^{r}\left((-n_j)_{\gamma k_j}\left(\frac{hy}{(-v)^{\gamma}}\right)^{k_j}\frac{1}{k_j!}\right)\right]$$

$$\frac{1}{(1+p-n_1-\ldots-n_r)_{\gamma(k_1+\ldots+k_r)}}I_1(k)+\sum_{m=1}^{\infty}\frac{\sinh\frac{2m\pi y}{a}\cos\frac{2m\pi x}{a}}{2^{\eta-1}\sinh\frac{m\pi b}{a}}$$

$$\frac{1}{(1+p-n_1-\ldots n_r)_{\gamma(k_1+\ldots+k_r)}}I(k) \qquad (4.30.3)$$

When $0<x<\frac{a}{2},0<y<\frac{b}{2}$, provided that all conditions of (4.26.4), (4.27.1) and (4.27.3) are satisfied.

The expansion formula is

$$\left(\cos\frac{\pi x}{a}\right)^{\eta}\frac{(-v)^{-n_1-\ldots-n_r}}{(-p)_{n_1+\ldots,+n_r}}\left[y\left(\cos\frac{\pi x}{a}\right)^{2\rho}\right]^{n_1/\gamma}\ldots\left[y\left(\cos\frac{\pi x}{a}\right)^{2\rho}\right]^{n_1/\gamma}$$

$$H_{n_1,\ldots n_r}^{(h,\gamma,v,p)}\left[\left[y\left(\cos\frac{\pi x}{a}\right)^{2\rho}\right]^{-r/\gamma}\right]I_{p_i,q_i:r}^{m,n}\left[z\left(\cos\frac{\pi x}{a}\right)^{2\sigma}\middle|\begin{matrix}A*\\B*\end{matrix}\right]$$

$$=\frac{1}{\sqrt{\pi}}\sum_{k_1=0}^{[n_1/\gamma]}\ldots\sum_{k_r=0}^{[n_r/\gamma]}\left[\prod_{j=1}^{r}\left((-n_j)_{\gamma k_j}\left(\frac{hy}{(-v)^{\gamma}}\right)^{k_j}\frac{1}{k_j!}\right)\right]\frac{1}{(1+p-n_1-\ldots-n_r)_{\gamma(k_1+\ldots+k_r)}}$$

$$I_1(k)+\sum_{m=1}^{\infty}\frac{\cos\frac{2m\pi x}{a}}{2^{\eta-1}}\sum_{k_1=0}^{[n_1/\gamma]}\cdots\sum_{k_r=0}^{[n_r/\gamma]}\left[\prod_{j=1}^{r}\left((-n_j)_{\gamma k_j}\left(\frac{hy}{(-v)^{\gamma}4^{\rho}}\right)^{k_j}\frac{1}{k_j!}\right)\right]$$

$$\frac{1}{(1+p-n_1-\ldots n_r)_{\gamma(k_1+\ldots+k_r)}}I(k) \qquad (4.30.4)$$

When $0<x<\frac{a}{2}$, provided that all conditions of (4.26.4), (4.27.1) and (4.27.3) are satisfied. Further, making an use of the result due to Chandel, Agarwal and Kumar (p. 27, eq. (1.4) and (1.5)).

$$\lim_{p\to\infty}H_{n_1,\ldots,n_r}^{(h,\gamma,1,p)}\left(\frac{x_1}{p},\ldots,\frac{x_r}{p}\right)=,\ \lim_{p\to\infty}H_{n_1,\ldots,n_r}^{(h,\gamma,1/p,p)}(x_1,\ldots,x_r)=g_{n_1}^{\gamma}(x_1,h)\ldots g_{n_r}^{\gamma}(x_r,h) \qquad (4.30.5)$$

To the results (4.30.2), (4.30.3) and (4.30.4), we get another different relation in similar way.

Further again, applying the relation

$$g_n^2(x,-1/4)=2^{-n}H_n(x) \qquad (4.30.6)$$

To the above results, we evaluate another result for Hermite polynomials by same techniques.

Other special cases and applications of our results may be obtained by making use of the work of Chandel and Sengar , Srivastava and Karlsson and Srivastava and Manocha , due to lack of space we omit them.

BIBLIOGRAPHY

Abdul-Halim, N. and Al-Salam, W.A (1963).; Double Euler transformation of certain hypergeometric functions, Duke Math. J., 30, 51-62.

Abiodum, R.F.A. and sharma, B. L. (1971).summation of series involving generalized hypergeometric functions of two variables Glasnik Math. Ser.III 6 (26), 253-264.

Agal, S. N. and Koul, C.L. (1981).On integrals involving the H -function of several variables.Jnanbha 11.

Agarwal, N. (1960). A q -analogue of MacRobert's generalized E -function, Ganita, 11:49-63.

Agrawal, R.K. (1980).A multiple integrals involving the H - function of two variables. Pure Appl. Math.Sci.12, 121-126.

Agrawal, R.K. (1980a).A Study of two-dimensional integral transforms and special sfunction involving one or more variables. Ph.D. Thesis, univ. of Rajasthan, India.

Agrawal, R.P. (1965). An extension of Meijer's G -function. Proc. Nat. Inst. Sci. India Part A, 31,536-546.

Agrawal, R.P. (1970).On certain transformation formulae and Meijer's G -function of two variables. Indian J Pure Appl.Math. 1, 537-551.

Agrawal, R.P. (1988).Glimpses of the theory of generalized hypergeometric functions. Ganita Sandesh Vol.2, No. 2, 87-93.

Anandani, P. (1969). Some integral involving generalized associated Legendre's function and the H-function .Proc. Nat. Acad. Sci. India Sect. A 39, 127-136.

Anandani, P. (1969a). On Some integrals involving generalized Legendre's associated function and the H -function .Proc. Nat. Acad. Sci. India Sect. A 39, 341-348.

Anandani, P. (1969b). Some integrals involving Product of generalised Legendre associated functions and the H -function. J. Sci. Engg. Res.13, 274-279.

Anandani, P. (1969c). Some expansion formulae for H -function 1V. Rend. Circ. Math. Palerma (2) 18, 197-214.

Anandani, P. (1970). Some expansion formula for the H -function involving associated Legendre's function. J. Natu. Sci.and Math. 10, 49-51.

Anderson, W.J., Haubold, H.J., Mathai, A.M. (1994). Astrophysical Thermonuclear Functions, Astrophys. Space Sci. 214, 49.

Arora, A.K., Raina, R.K. and Koul, C.L. (1985), On the two-dimensional Weyl fractional calculus associated with the Laplace transforms, C.R. Acad. Bulg. Sci., 38(2),179-182.

Aslam Chaudhry,M., Zubair , S.M.(2002). On a class of Incomplete Gamma Functions with Applications, Chapman and Hall/CRC, Boca Raton.

Aslam Choudhary M (1999). Transformation of the extended gamma functions $\Gamma_{0,2}^{2,0}[(B,X)]$ with applications to astrophysical thermonuclear functions, Astrophysics and Space Science, 262, vol. (2) 263-270.

Aslam Choudhary M and Zubair SM (2000). On a Class of Incomplete Gamma Functions with Applications, Chapman and Hall/CRC, New York.

Bajpai, S. D. And AL-Hawaj, A.Y. (1989A). Special functions and time-domain synthesis problem .The J. of the Indian Academy of Mathematics, Vol. 11, No. 1.

Bajpai, S.D. And AL-Hawaj, A. Y. (1989). Meijer's G-function and ring-shaped heat conductor. Ganit (J. Bangladesh Meth. Soc.) Vol. 9, No.1, 33-38.

Bajpai, S.D. (1968). Some expansions formulae for G-function involving Bessel functions. Proc. Indian Acad. Sci68 A, 285-290.

Bajpai, S.D. (1969). An expansion formula for fox's H-function. Proc. Cambridge philos. Soc. 65, 683-685.

Bajpai, S.D. (1969a). An expansion formula for H-function Involving Bessel functions. Labdev J. Sci. Tech. Part A 7, 18-20.

Bajpai, S.D. (1971). Some results involving Fox's H-function. Portugal Math. 30, 45-52.

Bajpai, S.D. (1980). An exponential Fourier for Fox's H-function. Math. Education 14, Sect. A, 32-34.

Bajpai, S.D. (1991). Fourier series and expansions for Fox's H-function of two variables and two-dimensional Heat equation. Hadronic Journal Val. 14, 195-201.

Bajpai, S.D. (1991a). Double and multiple half-range Fourier Series of Meijer's G-function. Acta Mathematica Vietnamica, Vol. 16, No. 1, 27-37.

Bajpai, S.D. (1991b). A new class of two-dimensional Expansion of Meijer's G-function involving Bessel Functions and Jacobi polynomials. Vijnana Parishad Anusandhan patrika, Vol. 34, No. 4, 233-236.

Bajpai, S.D. (1992). Atypical two-dimensional exponential Bessel expansion of Fox's H-function. The mathematics Education,vole. XXVI, NO. 1, 1-3.

Banerji, P.K. and Sethi, P.L.(1978). Operators of a generalized function of n- variables. The Mathematics Education, Vol. 16, No2, 152-159.

Barnes, E.W. (1908). A new development of the theory of the Hypergeometric functions. Proc. London Math. Soc. (2) 6, 141-177.

Basister, A.W. (1967). Transcental functions (Satisfying Non-homogeneous Linear differential equation). Macmillan, New York.

Bergstroem L, Iguri S and Rubinstein H (1999).Constraints on the variation of the fine structure constant from big bang nucleosynthesis, Physical Review D, 60 (1) 045005-1:9 **.**

Bhise, V.M. (1962). On the derivative of Meijer's G-function And Mac Robert's e-function. Proc. Nat. Acad. Sci. India, 32, 349-354.

Bhise, V.M. (1964). Some finite and infinite series of Meijer's Laplace transform. Math. Ann. 154, 267-272.

Bhise,V.M. (1963). Finite and infite series of Meijer's G-function and the multiplication formula for G -function. Jour. Indian Math. Soc. (N.S.) 27, 9-14.

Bora, S.L. and Kalla, S.L. (1970). Some results involving Generalized function of two variables. Jyungpook Math. J. 10, 133-140.

Bora, S.L. and Kalla, S.L. (1971). An expansion formula for the generalized function of two variables. Univ. Nac. Tucuman Rev. Ser. A21, 53-58.

Bora, S.L. and Saxena, R.K. (1971).Integrals involving Product of Bessel functions and generalized hyperGeometric functions. Publ. Inst. Math. (Beograd) (N.S.) 11 (25), 23-28.

Bora, S.L. Kalla, S.L. and Saxena, R.K. (1970). On integral Transforms. Univ. Mac. Tucuman Rev. Ser. A 20, 181-188.

Bora, S.L., Saxena, R.K. and Kalla, S.L. (1972). An expansion Formula for fox's H-function of two variables. Univ. Nac. Tucuman Rev. Ser. A 22, 43-48.

Boersma, J. (1962). On a function which is a special case of Meijer's G-function, Compositio Mathematica, 15, 34-63.

Braaksma, B.L.J. (1963). Asymptotic expansions and analytic Continuations for a class of Barnes-integrals. Composito Math. 15, 339-341.

Bromwich, T.J.I' A. (1931). An introduction to the theory of infinite series, 2nd ed. Macmillan and Co., London.

Bromwich, T.J.T.A. (1954). Theory of Infinite Series, Macmillan, London .

Brown RE and Jarmie N (1990). Differential cross sections at low energies for $^{2}H(d,p)^{3}H$ and $^{2}H(d,p)^{3}He$, Physical Review C, 41(3), 1391-1400.

Buschman, R.G. (1974). Partial derivatives of the H-function with respect to parameters expressed as finite sums and as integrals, Univ. Nac. Tucuman Rev. Ser., A24, 149-155.

Buschman, R.G. and Gupta, K.C. (1975). Contigous relations for the H-function of two variables, Indian J. Pure Appl. Math.,6,1416-1421.

Buchman, R.G. (1972). Contiguous relations are related Formulas for the H-function of Fox. Jnanabha Sect. A 2, 39-47.

Buchman, R.G. (1979). Integral equations and Laplace transformations. J. Indian Acad. Math. 1, No. 1, 1-4.

Buschman, R.G. and Srivastava, H.M. (1990). The $\overline{H}$ – function associated with a certain class of Feynman integrals, J. Phys. A: Math. Gen. 23, 4,707-710.

Buschmann RG and Srivastava HM (1990). R.G. The $\overline{H}$ -function associated with a certain class of Feynman integral, J.Phys.A:Math.Gen.23 (1), 4707-4710.

Chandel, R.C.S., Aharwal, R.D. and Kumar, H. (1990). A class of polynomials in

several variables, Ganita Sandesh, 1, 27-32.

Chandel, R.C.S., Aharwal, R.D. and Kumar, H. (1992). An integral involving sine functions , the Kempe de Feriet functions and the multivariable H -function of Srivastava and Panda and its applications in a potential problem on a circular disc, J. PAAMS XXXV (1-2), 59-69.

Chandel, R.C.S. and Sengar, S. (2002). On two boundary value problems, Jnanabha 31/32 (2002), 89-104.

Chaurasia, V.B.L. (1991). Applications of the multivariable H –function in heat conduction in a rod with one and at zero degree and the other and insulated, Jnanabha ,21(1991), 51-64.

Chaturvedi, K.K.and Royal, A.N. (1972). A*-function. Indian J. Pure Appl. Math. 3, 357-360.

Chaurasia, V.B.L and Agnihotri, A.(2010).Two dimensional generalized weyl fractional calculus and special functions. Tamkang journal of Mathematics, volume 41,No. 2,pp 139-148.

Chaurasia, V.B.L and Srivastava, A. (2006).Two dimensional generalized weyl fractional calculus pertaining to two dimensional $\overline{H}$ -transforms. Tamkang journal of Mathematics, volume 37,No. 3,pp 237-249.

Chaurasia, V.B.L. (1988). Integration of certain products Associated with the multivariable H -function. Gahnita Sandesh Vol. 2, No. 2, 66-69.

Chaurasia, V.B.L. and Tyagi, Sanjeev (1991). Integrals Involving generalized Lauricella's function and the H -function of several complex variables. Indian J. Pure Appl. Math. 22(5), 397-401.

Chauriasia, V.B.L. and Sharma , S.C. (1984); An integral involving extended Jacobi polynomials and H-function of several complex variables, Vijnana Parishad ,22, 267-272 .

Chiney, S.P. and Bhonsle, S.R. (1975); Some results involving extended Jacobi polynomials, Rev. Univ. Nac. , Tucuman, 25, 7-11.

Churchill, R.V. (1961). Fourier series and boundary value Problems. Mc-Grow Hill, New York.

Clayton DD (1983). Principles of Stellar Evoluation and Nucleosynthesis, Second Edition, The University of Chicago Press, Chicago and London.

Coraddu M, Kaniadakis G, Lavagno A, Lissia M, Mezzorani G and Quarati P (1999) Thermal distributions in stellar plasma, nuclear reactions and solar neutrinos, Brazilian Journal of Physics, 29(2),153-168.

Coraddu M, Lissia M, Mezzorani G and Quarati P (2003). Super-Kamiokande hep neutrino best fit: a possible signal of non-Maxwell an solar plasma, Physical A, 326 (2), 473-481.

Critchfield CL (1972). Analytic forms of thermonuclear function in Cosmology, Fusion and Other Matters: George Gamnow Memorial Volume, Ed. F. Reines, University of Colorado Press, Colorodo,, vol.34(2), 186-191.

Critchfield, C.L. (1972). In: Comology Fusion and Other Matters: George Gamow Memorial Volume, Ed. F. Reines, Colorado Associated University Press, Colorado, 186.

Davis Jr R (2003). A half-century with solar neutrinos, Reviews of Modern Physics, 75(1), 985-994.

Davis Jr., R. (2003). Nuclear and neutrino astrophysics, Rev. Mod. Phys. 75, 985.

Deshpande, V.L. (1971). On the derivatives of G-function of two variables and their applications, Proc. Nat. Acad. Sci. India, 41A, 60-68.

Deshpande, V.L. (1979). Finite sum representations of the H-function of two and more variables with respect to parameters, Indian J. Pure Appl. Math., 10, 1514-1524.

Doetsch, Gustav (1950). Handbuch der Laplace-transformation, Vol. I, Verlag Birkhauser, Basel (1950).

Detach, Gustav (1950). Handbuch der Laplace-transformation. Vol.1,Verlag Birkhauser, Basel.

Dighe, M. And Baishe, V.M. (1979/80). On composition of Fractional integral operators as an integral operator with flourier type kernel. Math. Notate 27, 237.

Ditkin, V.A. and prudnikov, A.P. (1962). Operational Calculus in two variables and its applications. Translated from the Russian by D.M.G. Wishart. International series of monographs on pure and Appl. Math. Vol, 24, Pergamon press, Oxford.

Dixit, A.K. (1981). On some integral relations involving the general H-function of several complex variables. Indian J. pure Appl. Math. 12, 977-983.

Doetseh, Gustav (1974). Introduction to the theory and applications of the Laplace transformation. Springer verlag, Berlin Heidelberg, New York.

Edwards, J(1954). A Treatise on Integral Calculus, Chelsea, New York .

Erdelyi, A. (1940). On fractional integration and its application of the theory of Hankel transform. Quart. J.Math. (Oxford) (1) 11, 293-303.

Erdelyi, A. (1950-51). On some functional transformations.Univ. Politec. Torino Rend. Sem. Mat. 10, 217-234.

Erdelyi, A. (1951). On a generalization of the Laplace transformation. Proc. Edinburgh Math. Soc. (2) 10, 53-55.

Erdelyi, A. (1953) Higher Transdental Functions, vol. I, Mc Graw- Hill, New York.

Erdelyi, A. (1954). Tables of Integral Transforms, vol.I, Mc-Graw Hill, NewYork.

Erdelyi, A. (1954).Higher Transcendental Functions, vol. I, II, McGraw-Hill, New York.

Erdelyi, A. (1975). Fractional integrals of generalized Functions, Fractional calculus and its applications. (B. Ross, ed.). Springer-verlag, Berlin, Heidelberg and NewYork.

Erdelyi, A. And kober, h. (1940). Some remarks on Hankel Transforms. Quart. J. Math. (Oxford) 11, 212-221.

Erdelyi, A., Magnus, W., Oberhettinger, F. And Tricomi, F.g. (1953). Higher transcendenatal functions, Vol. I and II, McGraw-Hill, New York.

Erdelyi, A., Magnus, W., Oberhettinger, F. And Tricomi, F.g. (1954). Tables of integral transform, Vols. I and II. MaGraw-Hill, New York.

Erdelyi, A., Magnus, W., Oberhettinger, F. And Tricomi, F.G. (1955). Higher transcendental functions, Vol. III. McGraw-Hill, New York.

Exton, Harold (1976). Multiple hypergeometric functions and applications. Ellis Horwood Ltd., Chichester.

Exton, Harold (1978). Handbook of hypergeometric integrals. Ellis Horwood Ltd., Chichester.

Ferreira C. and Lopez J.L. (2004). Analytic expansions of thermonuclear reaction rates, Journal of Physics A: Mathematical and General, 37(4), 2637-2659.

Fowler, W.A (1984). Experimental and theoretical nuclear astrophysics: the quest for the origin of the elements, Reviews of Modern Physics, 56(2), 149-179.

Fowler, W.A. (1984). Maxwell-Boltzmann velocity distribution, Rev. Mod. Phys., 56, 149.

Fox, C (1961). The G and H function as symmetric Fourier kernels Trans. Amer. Math. Soc. 98, 395-429.

Fox, C. (1928). The asymptotic expansion of generalized Hypergeometric functions. Proc. London Math. Soc. (2) 27, 389-400.

Fox, C. (1961).; The G and H function as symmetric Fourier kernels Trans. Amer. Math. Soc. 98, 395-429.

Fox, C. (1963). Integral transforms based upon fractional Integration. Proc. Cambridge philos. Soc. 59, 63-71.

Fox, C. (1965). A formal solution of certain dual integral Equations. Trans. Amer. Math. Soc. 119, 389-398.

Fox, C. (1971). Solving integral equations by L and L^{-1} operators. Proc. Amer. Math. Soc. 29, 299-306.

Fox, C. (1972). Application of Laplace transforms and their Inverses. Proc. Amer. Math. Soc. 35, 193-200.

Fujiwara, I. (1966); A unified presentation of classical orthogonal polynomials, Math. Japan, 11, 133-148.

Garg, A. AND Singh, Y. (2009), Some properties of generalized Voigt Function, 214Ultra Scientist Vol. 21(1) M , 211- 214.

Garg, A., Singh, Y., and Ramawat, A. (2004), Application of Bessel polynomials in the study of time domain synthesis problems, The Math. Education, vol.38 (3), 148-152.

Garg, M. (1980). A note on a generating function for the multivariable H-function. Jnanabha (professor Arthus Erdelyi Memorial Volume), 9/10.

Garg, R.S. (1981). Some general integral relations for the multivariable H-function. Jnanabha.

Garding , (1947). The solution of Cauchy's problem for two totally hyperbolic linear differential equation by means of Reisz integrals, Ann. Of Math. 48, 785-826.

Gasper, G. and Rahman, M. (1990). Basic Hypergeometric Series, Cambridge University Press, Cambridge.

Gautam, A.N. (1981). On the A-function, Vijnana Parishad Anusandhan Patrika, vol. 23(4), 77-81.

Gautam, G.P. and Goyal, A.N. (1981). On Fourier kernels and asymptotic expansion. Indian J. Pure Appl. Math. 12, 1109-1105.

Gautam, G.P. and Goyal, A.N. (1983). On a multivariable A-function. Univ. Nac. Tucuman REV. Ser. A 21.

Gell-Mann M and Tsallis C (2004). (Eds.); NNonextensive Entropy: Interdisciplinary Applications, Oxford University Press, New York.

Gokhroo, D.C. (1974). Sum of an infinite series involving meijer's G-function. Proc. Nat. Acad. Sci. India44 (A) I. 65-68.

Goyal, S.P. (1970). On some finite integrals involving generalized G -function. Proc. Nat. Acad. Sci. IndiaSect. A 40, 219-228.

Goyal, S.P. and Agarwal, R.K. (1980).Basic series relations for the H-function of two variables. Pure Appl. Math. Sci. 11, 39-48.

Goyal, S.P. and Agawal, R.K. (1980a). A two-dimensional Neumann expansion for the H-function of two variables. Kyungpook Math. J. 20, 95-103.

Goyal, S.P. and Agrawal, R.K. (1980b). The distributions of a linear combination and the ratio of product of random variables associated with the multivariable H-function. Jnanabha (professor Arthur Erdelyi Memorial Volume) 9/10.

Goyal, S.P. and Agrawal, R.K. (1981). Bivariate distributions and multivariable H-function. Indian J. Pure Appl. Math.12, 380-387.

Goyal, S.P. and Jain, R.N. (1987). Fractional integral operatiors and the generalized hypergeometric functions. Indian J. Pure Appl. Math. 18, 6, 251-259.

Goyal, S.P., Jain, R.N. and Gaur, Neelima (1991). Fractional integral operators involving a product of generalized hypergeometric functions and a general class of polynomials. Indian J. Pure Appl. Math. 22(5), 403-411.

Gradshteyn, I.S. and Ryzhik, I.M. (1963). Tables of Integrals, Series and Products, Academic Press, NewYork.

Gradsteyn, I.S. and Ryzhik, I. M. (1980). Tables of Integrals, Series and Products. Corrected and Enlarged Eddition, Academic Press. Inc.

Gupta, K.C. (1965). On the H-function. Ann. Soc. Sci.Bruxlles Ser. I. 79, 97-106.

Gupta, K.C. (1966). Integrals involving the H-functions. Proc. Nat. Acad. Sci. India Sect. A 36, 504-509.

Gupta, K.C. and Agarwal, S.M.(1989); Fractional integral formulae involving a general class of polynomials and the multivariable H function, Proc. Indian Acad. Sci. (Math. Sci) 99, 169-173.

Gupta, K.C. and Agrawal, S.M. (1989). A finite integral involving a general class of polynomials and the multivariable H-function. Indian J. Pure Appl. Math. 20(6), 604-608.

Gupta, K.C. and Garg, Mridula (1984). A study of some multi- dimensional fractional integral operators. Jnanabha Vol. 14, 53-69.

Gupta, K.C. and Jain, U.C. (1966). The H-function-II. Proc. Nat. Acad. Sci. India Sect. A 36, 594-609.

Gupta, K.C. , Goyal, S.P. and Handa, S. (1976). Fractional integral operators involving two variables, Univ. Nac. Tucuman Rev. Ser. A 26 , 159-171.

Gupta, K.C. and Jain, U.C. (1968).On the derivative of the H-function. Proc. Nat. Acad. Sci. India Sect. A 38, 189-192.

Gupta, K.C. and Jain, U.C. (1969). The H-function - Iv. Vijnana parishad Anusandhan patrika 12, 25-30.

Gupta, K.C. and Koul, C.L. (1977): An integral involving H-function, Math. Student, 45, 33-38.

Gupta, K.C. and Mittal, P.K. (1970). The H-function transform. J. Austral. Math. Soc. 11, 142-148.

Gupta, K.C. and Mittal, P.K. (1971). The H-function transform. II. J. Austral. Math. Soc. 12, 444-450.

Gupta, K.C. and Soni, R.C. (2002); A study of H-functions of one and several variables, J. Rajasthan Acad. Phys. Sci. 1, 89-94.

Gupta, K.C. and Srivastava, A.(1972). On finite expansions for the H-function. Indian J. Pure Appl. Math. 3, 322-328.

Gupta, K.C., Agarwal, S.M. and Soni, R.C. (1990); Fractional integral formulae involving the multivariable H-function and a general class of polynomials, Indian J. Pure Appl. Math. 21(1990), 70-77.

Gupta, P.N. and Rathie, P.N. (1968). Some results involving generalized hypergeometric functions. Rev. Mat. Univ. Purma (2) 9, 91-96.

Gupta, R.K., Saxena, R. K. (1992). On certain transformations of multiple basic Hypergeometric functions J. Math. Phy. Sci., 26, no.4, 413- 418.

Gupta, Rajni (1988). Fractional integral operators and a general class of polynomials. J. Indian Acad. Math. Vol. 10, No. 1, 32-37.

Gupta, Rajni (1988). Transformation formulas for the multivariable H- function, Indian J. Of Mathematics, Vol. 30, N0.2,105-118.

Gupta, Rajni (1990). Fractional integraloperators and a general class of polynomials. Indian J. of Mathematics Vol. 32, No. 1, 69-77.

Gupta,K.C. , Jain, Rashmi and Sharma,Arti (2003). A study of unified finite integral with applications, J.Rajashthan Acad. Phys. Sci. Vol.2,No.4, 269-282.

Handa, S. (1976). A study of generalized functions of one and two variables,Indian J. of Math., 11-16.

Hardy, G.H. and Littlewood, J.E. (1925). Some properties of fractional integrals. Proc. London Math. Soc. (2), 24, 37-41.

Hathie, N.R. (1979). Integrals involving H-function. Vijnana Parishad Anusandhan Patrika 22, 253-258.

Hathie, N.R. (1980). Integrals involving H-function. II. Vijnana Parishad Anusandhan Patrika 23, 331-337.

Haubold HJ and Mathai AM (1984). On nuclear reaction rate theory, Annalen der Physic (Leipzig), 41(1), 380-396.

Haubold HJ and Mathai AM (1998). An integral arising frequently in astronomy and physics, SIAM Review**,** 40(2), 995-997.

Haubold, H.J. and Mathai, A.M. (1984). On the nuclear energy generation rate in a simple analytic stellar mode, Annalen der Physik, 41(6), 372-379.

Haubold, H.J. and Mathai, A.M. (1985). The Maxwell-Boltzmann approach to the nuclear reaction rate theory, Forschr. Phys., 33, 623-644.

Haubold, H.J. and Mathai, A.M. (1986c). Analytic results for screened non-resonant nuclear reaction rates, Astrophysics and Space Science, 127, 45-53.

Haubold, H.J. and Mathai, A.M.(1984a). On nuclear reaction rate theory, Annalen der Physik, 41(6), 380-396.

Haubold, H.J. and Mathai, A.M.(2000). The fractional Kinetic equation and thermonuclear functions, Astrophysics and Space Science 273, 53-63.

Hussein MS and Pato MP (1997). Uniform expansion of the thermonuclear reaction rate formula, Brazilian Journal of Physics, 27(2), 364-372.

Inayat-Hussian,A.A. (1987). New properties of hypergeometric series derivable from Feynman Integrals II. A generalization of the H-function. J Phy. A: Math. 20, 4119-4128.

Jain, R. (1992): A study of multidimensional Laplace transform and its applications, Soochow J.of Math., 183-193.

Jain, R.N (1965). Some double integral transformations of certain hypergeometric functions, Math. Japon, 10, 17-26.

Jain, R.N. (1966). A finite series of G-function. Math. Japan, 11, 129-131.

Jain, R.N. (1969). General series involving H-functions. Proc. Cambridge philos. Soc. 65, 461-465.

Jain, U.C. (1967). Certain recurrence relations for the H-function. Proc. Nat. Inat. Sci. India parta 33, 19-24.

Jain, U.C. (1968). On an integral involving the H-function. J. Austral. Math. Soc. 8, 373-376.

Joshi, C.M. and Arya, J.P. (1981). Concerning some properties of the H-function of several variables. Indian J. Pure Appl. Math. 12, 826-843.

Kalla, S.L. (1966).Some theorems on fractional integration. Proc. Nat. Acad. Sci. India Sect. A 36, 1007-1012.

Kalla, S.L. (1969). Integral operators involving Fox's H-function. Acat. Maxicana Ci. Tecn. 3, 117-122.

Kalla, S.L. (1969).Some theorems on fractional integration II. Proc. Nat. Acad. Sci. India Sect. A 39, 44-56.

Kalla, S.L. (1970). Fractional integration operators involving generalized hypergeometric functions. Univ. Nac. Tucuman Rev. Ser. A. 20, 93-100.

Kalla, S.L. (1970-71). On operators of fractional integration. Math. Notae 22, 89-93.

Kalla, S.L. (1976).On operators of fractional integration. II. Math. Notae 25, 29-35.

Kalla,S.L. and kiryakova, V.S. (1990). An H-function generalized fractional calculus based upon compositions of Erdelyi-kober operators in L_p. Math. Japan. 35, 1151-1171.

Khadia, S.S. and Goyal, A.N. (1970). On the generalized function of $'n'$ variables. II. Vijnana parishad Anusandhan patrika 13, 191-201.

Kim,Y.C. and Srivastava,H.M.(1998).Fractional integral and other linear operators associated with the Gaussian Hypergeometric function.Complex variable theory Appl.34.293-312.

Kiryakova, V.S. (1986).On operators of fractional integration involving Meijer's G -function, Compt. Acad. Bulg. Sci. 39, 25-28.

Kiryakova, V.S. (1988). Fractional integration operators involving Fox's $H_{m,m}^{m,0}$ -function. Compt. Rend. Bulg. Sci. 41, 11-14.

Kober, H. (1940). On fractional integrals and derivatives, Quart. J. Math. Oxford. Ser. 11, 193-211.

Korganoff, V. (1980). Introduction to Astrophysics, Reidel Publication, Boston.

Kraetzel, E. (1979). in: Generalized Functions and Operational Calculus, Proc. Intern. Conf. Varna, 1975, Bulg. Acad. Sci., Sofia, 148.

Kumar, H. (1992-1993). Special functions and their applications in modern science and technology, Ph. D. thesis, Barkathullah University, Bhapol, M.P., India.

Kumbhat, R.K. and Saxena R.K. (1975). Theorems connecting L , L^{-1} and fractional integration operators. Proc. Nat. Acad. Sci. India 45(A) III, 205-209.

Kumbhat, R.K. and Saxena, R.K. (1974). A formal solution of certain triple integral equations involving H -functions, Proc. Nat. Acad. Sci. India, Sect. A 44, 153-159.

Lavagno A and Quarati P. (2006). Metastability of electron-nuclear astrophysical plasmas: motivations, signals and conditions, Astrophysics and Space Science, 305(6), 253-259.

Lawrynowicz, J. (1969). Remarks on the preceding paper of P .Anandani. Ann. Polon. Math. 21, 120-123.

Lebedev, V.N. (1965). Special functions and their applications. (Translated from the Russian by Richard A. Silverman) Prentice-Hall, New Jersey.

Lerch, E. (1903). Sur UN. Point de la theories des functions generatrices d' Abel, Acta Math. 27, 339.

Lin, S.-D. , Liu, S.-J. and Srivastava, H. M. (2011), Some families of hypergeometric polynomials and associated multiple integral representations, Integral Transforms Spec. Funct. 22. 403-414.

Loonker,D. and Banerjee P.K.(2004).Distributional Laplace-Hankel transform by fractional integral operators. The Mathematics student.vol 73 , No.1-4.

Love, E.R. (1970). Changing the order of integration. J. Austral. Math. Soc. 11, 421-432.

Love, E.R.(1967). Some integral equations involving hypergeometric functions. Edinburg Math. Soc. (3) 15, 169-198.

Luke, Y.L. (1975). Mathematical functions and their approximation. Academic press, New York.

Mathai A.M. and Saxena , R.K.(1978); The H -function with Applications in Statistics and Other Disciplines, John Wiley & Sons, New Delhi.

Mathai AM and Haubold HJ (2002). Review of mathematical techniques applicable in astrophysical reaction rate theory, Astrophysics and Space Science, 282 (8), 265-280.

Mathai AM and Haubold HJ (2007). Pathway model, superstatics, Tsallis statistics and a generalized measure of entropy, Physic A 375(8), 110-122.

Mathai AM and Haubold H. J. (1988): Modern Problems in Nuclear and Neutrino Astrophysics, Akademie-Verlag, Berlin**.**

Mathai, A.M. and Saxena, R.K. (1966). On a generalized hypergeometric distribution, Matrika, 11, 127-132.

Mathai, A.M. and Saxena, R.K. (1966). Tables of the generalized

hypergeometric distribution, Matrika, 11,163-169.

Mathai, A.M. and Saxena, R.K. (1971). A generalized probability Distribution,Sepatata de la revista mathematica Y Fisica Teorica, vol. 21,193-202.

Mathai, A.M. (1995). Special function of matrix argument II, Proc. Nat. Acad. Sci. India 35, Sect. A, 367-393.

Mathai, A.M. and Saxena , R.K. (1973). Gneralized Hypergeometric Functions

With Applications in Statistics and Physical Sciences, Spring-verleg, Lecture NotesNo. 348, Heidelberg .

Mathai, A.M. and Saxena, R.K. (1978). A generalized probability distribution,

Univ. Nac. Tucuman Rev. Ser. A21, 193-202.

Mathai AM and Saxena RK (1973). Generalized Hypergeometric Functions with Applications in Statistics and Physical Sciences, Springer-Verlag, Lecture Notes in Mathematics 348 Berlin, Heidelberg, New York.

Mathai, A.M. and Saxena, R.K. (1973). Generalized hypergeometric functions with applications in statistics and physical Sciences, Springer-Verlag, Lecture Notes No. 348, Heidelberg and New York.

Mathai, A.M. and Saxena, R.K. (1978). The H-function With Applications in Statistics and Other Disciplines, John Wiley and Sons. Inc. New York.

Mathai,A.M., Saxena,R.K., Haubold,H.J.(2004).Astrophysical Thermonuclear functions for Boltzmann-Gibbs statistics and Tsallis statistics,Physica A 344,649-656.

Mathur, S.L.(1974). Astudy of boundary value problems and special functions, Proc. Acad. Sci India Sect.A 43,56-61.

Mathur, B.L. (1981).Some results concerning a special function of several complex variables. Indian J.Pure Appl. Math. 12, 1001- 1006.

Mathur, B.L. and Krishna, S. (1977). On multivariate fractional integration operators. Indian J. Pure Appl. Maths. 8, 1078-1082.

McBride, A.c. and Roach, G.F. (1985). Fractional calculus. Pitman Advanced publishing program .

McLachlan, N.W. (1962). Laplace Transforms and Their Applications to Differential Equations, Dover Publications, New York.

Meijer, C.S. (1964). On the G-function, I. Proc. Nederl. Akad. Wetensch. 49, 227-237.

Miller, K.S. (1975).The Weyl fractional calculus,(Lecture Notes in Math. 457), Berlin 80-89.

Mishra, A.P. (1975).On a fractional differential operator, Ganita 26(2) ,1-18

Mishra, S. (1990). Integrals involving expontial function, generalized hypergeometric series and Fox's H-function and Fourier series for products of generalized hypergeometric functions,J. Indian Acad. Math.,12(1), 33-47.

Mittal, P.K. and Gupta, K.C. (1972). An integral involving generalized function of two variables. Proc. Indian Acad. Sci. Sect. A 75, 117-123.

Mohammad (1987). Special Functions and its Applications, Ph.D. Thesis, Calcutta University.

Munot, P.C. and Kalla, S.L. (1971). On an extension of generealized H-function of two variables. Univ. Nac. Tucuman Rev. Ser. A 21, 67-84.

Munot, P.C. and Kalla, S.L.; On an extension of generalized function of two variables, Universidad Nac.De. Tuc. Rev. Ser. A(1971),21-27.

Munot, P.C. and Mathur, R. (1983). Certain integral relations for the multivariable H-function. Indian J. Pure Appl. Math. 14(8), 955-964.

Nair, V.C. (1969); Investigations in Transform Calculus, Ph.D. Thesis, University of Rajasthan.

Nair, V.C. and Nambudripad, B.N. (1973). Integration of H-function with respect to their parameters. Proc. Nat. Acad. Sci. India, 43(A), IV.

Nair, V.C. and Samar, H.S. (1971A). The product of two H-function expressed as a finite integral of the sum of a series of H-functions. Math. Education Sect.A5,45-48.

Nair, V.C. and Samar, M.S. (1971). An integral involving the product of three H-functions. Math. Nachr. 49, 101-105.

Nair, V.C. and Samar, M.S. (1975). A relation between the Laplace transform and the Mellin transform with applications, Portugaliae Mathematica, vol.34, Fasc. 3,149-155.

Panchal, M.K., Singh, M.P. and Chandramouli, A.B. (2008); Steady and unsteady incompressible fluid through porous medium, The Mathematics Education, vol.XLII (3),188-191.

Panda, R. (1977). Certiain integrals involving the H -function of several variables. Publ. Inst. Math. (Beograd) (N.S.), 22 (36), 207-210.

Panda, R. (1977a). Integration of certain products associated with the H-function of several complex variables. Comment. Math. Univ. St. Paul. 26, 115-123.

Panda, R. (1977b). On a multiple integral involving the H-function of several complex variables. Indian J. Math., 19, 157-162.

Pandey, R.N. and Pandey, S.K. (1985). Power series expansions of H-function of n-variables. Vijnana parished Anusandhan patrika 28(2), 137-149.

Paraser, B.P.(1968). Domain and range of fractional integration operators, Math. Japan, 12, 141-145.

Paris, R.B. and Kaminski,D. (2001). Asymptotic and Mellin-Barnes Integrals (Cambridge: Cambridge University Press).

Pasad, Y.N. (1986); On a multivariable I-function, Vijnana Parishad Anuusandhan Patrika, vol.29 (4), 231-235.

Pathak, R.S. (1970). Some results involving G- and H-functions. Bull.cal. Math. Soc. 62, 97.

Pathak, R.S. and Pandey, R.N. (1989). On certain fractional integral operators. Bull. Cal. Math. Soc. 81, 17-24.

Pathan, A.M. (1968). Certain recurrence relations. Proc. Cambridge Phills. Soc. 64,

1045-1048.

Prasad, Y.N. (1986). Multivariable I –function. Vijnana Parishad Anusandhan Patrika, 29, 232-235.

Prasad, Y.N. and Singh, S.N. (1977). Multiple integrals involving the generalized function of two variables. Ann. Fac. Sic. Univ. Nat. Zaire (Kinshasa) Sect. Math.Phys. 3, 252-265.

Ragab,F.M. (1957).Integration of certain fractional integral operators. Proc. Glasgow Math. Assoc. 3, 94-98.

Raina, R.K. (1984). On composition of certain fractional integral operators. Indian J. Pure Appl. Math. 15, 509-516.

Raina, R.K., Kiryakova,V.S. (1983). On the Weyl fractional operator of two dimensions, C.R. Acad. Bulg. Sci., 36(10),1273-1276.

Rainville, E.D. (1963). Special functions. The Macmillan co.Inc, New York.

Rainville, E.D. (1963).The Laplace transform: An introduction, The Macmillan Co., New York.

Ramawat, A., Singh, Y. and Saxena ,R. K. (1992). Multidimensional fractional integration operators associated with multivariable I-function. Ganita Sandesh 6 , 105-112.

Rao, C.R. (1965). Linear Statistical Inference and Its Applications, Wiley, NewYork .

Rathie, A.K. (1997). A new generalization of generalized hypergeometric functions, Le Mathematiche Fasc. II, 52, 297-310.

Riesz ,M (1949).L integrale de Riemann-Liouville et. le problem de Cauchy pour 'L' equations desondens, Societe Mathematique de France, Conference de La Re'Union international des Mathematiciens tenue a Paris on Juillet, Paris .

Reed, I.S. (1971) The Mellin type of double integral, Duke Math. J., 17, 565-572.

Riemann, B. (1876). Gesammelte Werke, pp. 331-344.

Rhodes, D.R. (1970). On the spheroidal functions, Jour. Res. Nat. Bureau of Standrads-73, Math. Sci. 74B, 187-209.

Saigo M. (1984). A generalization of fractional calculus, Fractional Calculus, Research Notes in Maths. 138, pitman.

Saigo, M and Raina, R.K. (1988); Fractional calculus operators associated with general class polynomials, Fukuoka Univ. Reports 18, 15-22.

Saigo, M.,Ram J. and Saxena R. K. (1995). On the two dimensional Weyl fractional Calculus associated with two dimensional H-transform, J. Fract. Calc. 8, 1995, pp.63-73.

Saigo, M.,Ram, J. and Saxena ,R. K. (1992). On the fractional calculus operator associated with H-function, Ganita Sandesh 636-47.

Saigo, M.A. (1978); A remark on integral operators involving the Gauss hypergeometric functions, Math. Rep. Kyushu. Univ. 11, 135-143.

Samko, S., Kilbas, A. and Maricher, O.I. (1987). Integrals and derivatives of fractional order and some of their applications (In Russian). Nauka i Technika, Minsk.

Saxena , V.P. (1982) : Formal solution of certain new pair of dual integral equations involving H-functions , Proc. Nat. Acad. Sci. India , A52, 366-375 .

Saxena R. K., Singh, Y. and Ramawat, A.(1993). On compositions of generalized fractional integrals. Hadronic J. Supplement 8, 137-159.

Saxena RK, Mathai AM and Haubold HJ. (2004): Astrophysical thermonuclear functions for Boltzmann-Gibbs statistics and Tsallis statistics, Physic A, 344 (5), 649-656.

Saxena, R. K. and Gupta, O.P. (1993).On certain relations connecting Erd´elyi-Kober operators and generalized Laplace transform. J. Fract. Calc. 3, pp.73-79.

Saxena, R. K., Ram, C. and Dave, O. P. (1994). Integrals associated with Gauss' hypergeometric series, Multivariable H-function an a general class of polynomials, Jnanabha, 14, pp35-48.

Saxena, R. K., Ram, C. and Dudi, N.(2005). Some results on a new generalized hypergeometric function, Acta Ciencia Indica, Math.Vol. 31 M 1265-1272.

Saxena, R. K., Ram, C.(1990). An integral involving multivariable H-function. Vijnana Parishad Anusandhan Patrika, 33, No.2, pp.87-93.

Saxena, R. K., Ram, J., Chandak, S.and Kalla, and S.L. (2008).Unified fractional integral formulas for the Fox-Wright generalized hypergeometric function, Kuwait J. Sci. Eng. 351-20.

Saxena, R. K., Ram, J.and Chandak, S. (2007). Unified fractional integral formulas involving the I-function associated with the modified Saigo operator, Acta Ciencia India Math. Vol 33M, No. 3, pp 693-704.

Saxena, R. K., Ram, J.and Suthar, D.L. (2005). Integral Formulas for the H-function Generalized Fractional Calculus associated with Erd´elyi-Kober operator of Weyl type, Acta Ciencia Indica, Math.Vol. XXXI M, No.3, 761-766.

Saxena, R. K., Ram, J.and Suthar, D.L. (2006).Certain properties of generalized fractional integral operators associated with Mellin and Laplace transformations, J.Indian Acad. Math. Vol.28, No.1, pp.166-177.

Saxena, R. K., Singh, Y. (1991). Integral operators involving Multivariable I-function. Proc. Nat. Acad. Sci. India,61(A), no.2, pp. 177-183.

Saxena, R.K. (1960), Some theorems on generalized Laplace transforms, Proc. Nat. Inst. Sc. India, 26 A400-413.

Saxena, R.K. (1960). An integral involving G - function. Proc. Nat. Inst. Sci. India Part A 26, 661-664.

Saxena, R.K. (1966). An inversion formula for a kernel involving a Mellin-Barnes type integral. Amer. Math. Soc.17, 771-779.

Saxena, R.K. (1967). On fractional integration operators, Math. Zeitcher, 96, 288-291.

Saxena, R.K. (1967). A formal solution of certain dual integral equations involving H-function. Proc. Cambridge Philos. Soc. 63, 171-178.

Saxena, R.K. (1971). Integrals of products of H-functions. Univ. Nac. Tucuman Rev. Ser. A 21, 185-191.

Saxena, R.K. (1971a). Definite integrals involving Fox's H–function. Acta Maxicana Ci. Tecn. 5, 6-11.

Saxena, R.K. and Kumar, R. (1990). Recurrence relations for the basic analogue of the H-function, J. Nat. Acad. Math. 8: 48-54.

Saxena, R.K. and Kumbhat, R. K. (1974). Integral operators' involving H-function. Indian J. Pure Appl. Maths. 5, 1-6.

Saxena, R.K. and Kumbhat, R.K. (1973). A generalization of the kober operators. Vijnana parishad Anusandhan patrika 16, 31-36.

Saxena, R.K. and Kumbhat, R.K. (1973a). Fractional integral operator of two variables. Proc. Indian Acad. Sci. Sect. A 78, 177-186.

Saxena, R.K. and Kumbhat, R.K. (1975). Some properties of generalized kober operators. Vijnana parishad Anusandhan patrika 18, 139-150.

Saxena, R.K. and Mathur, S.N. (1971). A finite series of the H-function. Univ. Nac. Tucuman Rev. Ser. A 21, 49-52.

Saxena, R.K. and Modi, G.C. (1974).Some expansions involving H-function of two variables. C.R. Acad. Bulgare Sci. 27, 165-168.

Saxena, R.K. and Modi, G.C. (1974a). Expansion formulae of the generalized H-function. Vijnana parishad Anusandhan patrika 17, 185-195.

Saxena, R.K. and Modi, G.C. (1975). Generalized H-function as a symmetrical Fourier kernel. Univ. Nac. Tucuman Rev. Ser. A, 25.

Saxena, R.K. and Singh, Y. (1991), A finite series for the multivariable A-function , Vijnana Parishad Anusandhan Patrika,vol. 34 No.3, 139-145.

Saxena, R.K. and Singh, Y. (1991), Integral operators involving multivariable I-function, Proc. Nat. Acad. Sci.India, 61(1), 11, 177-183.

Saxena, R.K. and Singh, Y. (1991), An expansion formula for multivariable A- function, Vijnana Parishad Anusandhan Patrika, vol. 34 No.4 197-205.

Saxena, R.K. and Singh, Y. (1993), On the derivative of the multivariable I-function, Vijnana Parishad Anusandhan Patrika, vol. 36 No.2, 93-98.

Saxena, R.K. and Singh, Y. (1993), Integral operators involving generalized H-function, Indian J. of Math. , 35,177-188.

Saxena, R.K. and Singh, Y. (1993), Integration of generalized H-function with respect to their parameters, Vijnana Parishad Anusandhan Patrika, vol. 36,(1),55-62.

Saxena, R.K., Singh, Y. and Ramawat, A. (1993), On compositions of generalized fractional integration, Hadronic journal supplement, (1993) 137-159.

Saxena, V.P. (1982): Formal solutions of certain new pair of dual integral equations involving H-functions, Proc.Nat.Acad.Sci. India A (52), 366-375.

Sethi, P.L. and Mahmoud,A.A. (2003) ; Altayeb; Fractional integral operators involving Fox's H -function for real positive definite symmetric matrix, J. Raj. Acad. Phy. Sci., vol.2 (2), 139-152.

Shah, M. (1969). Some results on Fourier series for H -functions. J. Natur. Sci. and Math. 9, 121-131.

Shah, M. (1973). Application of Hermite polynomials for certain properties of fox's H -function of two variables. Univ. Nac. Tucuman Rev. Ser. A 23, 165-178.

Sharma, B.L. (1965). On a generalized function of two variables. I. Ann. Soc. Bruxeles Ser. I, 79, 26-40.

Sharma, B.L. (1968). A new expansion formula for hypergeometric functions of two variables. Proc. Cambridge philos. Soc. 64, 413-416.

Sharma, K.C.and Singh, Roshni.(2010).Finite summation formulae for multivariable H-function. Tamkang journal of Mathematics, volume 41,No. 4,pp 375-377.

Sharma, O.P. (1965). Some finite and infinite integrals involving H -function and Gauss's hypergeometric functions. Collect. Math.17, 197-209.

Siddiqui, A. (1979). Integral involving the H -function of several variables and an integral function of two complex variables. Bull. Inst. Math. Acad. Sinica 7, 329-332.

Singh, R.P (1965). A note on double transformation of certain hypergeometric functions, Proc. Edinburgh. Math. Soc. (2),1491, 221-227.

Singh, Ratan (1968). On some results involving H -function of fox. Proc. Nat. Acad. Sci. India Sect. A 38, 240- 322.

Singh, Ratan (1970). An inversion formula for fox H -transform. Proc. Nat. Acad. Sci. India Sect. A, 40, 57-64.

Singh,Y.,Kamarujjama,M. and Khan,N.A. (2006). Integrals and Fourier series involving H-function, International J. of Mathematics and Analysis 1(1), 53-67.

Singh, Y, Kamrujjma, M. and Khan, N.A. (2008), Unified probability density function associated with general class of polynomials and Lauricella confluent hypergeometric function of several variables; International J. of Appl. Math and Eng. Sci., vol. 2 (1), 35- 45.

Singh, Y., Kamarujjma, M. and Khan. N.A. (2009), On Some finite integrals involving generalized polynomial sets and the multivariable H- function with applications, Journal Purvanchal Academy of Science, vol.15, 272-286.

Singh, Y, Kamrujjma, M. and Khan, N.A. (2010), N-fractional differential operators involving I-function, The South East Asian Bull. Of Math. vol.34, 181-184.

Singh, Y. (2004), An exponential Fourier series for the H-function, Ultra Science, vol.16 (1), 107-108.

Singh, Y. (2008), Fourier Series Involving The H-function, Journal of Research, vol.19(2),53-65.

Singh, Y. (2009), A general class of multivariate distribution involving H-function, Indian J. of Sci. & Tech., vol. (6), 1-3.

Singh, Y. (2009), A unified study of bicomplex spaces and functions, Indian J. of Sci. & Tech., vol. 7, 1-7.

Singh, Y. (2010), On some transformation formulas of H-function, International journal of computer *science and Emerging Technology, vol. 1(3), 92-97.*

Singh,Y.(2011),On Some Expansion Formulae Involving A Basic Analogue Of Generalized Hypergeometrion Function, Proceedings of Pakistan academy of sciences, 48(2), 135-139.

Singh. Y. (2005), On some convolution theorem involving H-function, Ultra Science, vol. 17 (1), 9-14.

Singh, Y.(2010), The Relation between Double Mellin Transform and Double Hankel Transform with Applications, Proc. Pakistan Acad. Sci. 47(2),107-111.

Singh, Y. (2012), A phenomenon in Euclid and non-euclidean geometric analysis, The Mathematics Education, vol.XLVI (3), 11-15.

Singh, Y. and Garg A. (2003), Application of Bessel polynomials with H-function in time domain synthesis problem, Ganita Sandesh, vol.17 (2), 41-44.

Singh, Y. and Garg, A. (2003), Double and multiple half- range Fourier series of the H-function, J. Raj. Acad. Phy.Sci., vol.2, (4), 251-256.

Singh, Y. and Garg, A. (2004), Application of generalized prolate spheroidal wave function in a linear flow of heat in an anisotropic material with H-function, Ganita Sandesh, vol.18 (2), 141-148.

Singh, Y. and Garg, A. (2007), On the derivative of H-function; Soochow J.of Math., vol.33, 1-4.

Singh, Y.and Garg A. (2008), A., Some finite integrals involving the product of Srivastava's polynomials and a certain H-function with applications, Kyungpook Journal of Math., vol.48(1), 165-171.

Singh,Y. and Khan, M.A. (2007), Some Integrals Involving generalized Prolate Spheroidal Wave Function and I - function with Application, Journal of Research(Science) ,vol.18(3),177- 183.

Singh, Y. and Khan, N.A.(2011), A unified study of fractional kinetic equation and thermonuclear functions, Advances in Applied Science research, 2(4), 383-390.

Singh, Y. and Mandia Harmendra (2011), A study of unified theorems involving the Laplace transforms with applications, International Journal of Research and Reviews in Computer Sciences, vol. 2(6), 1319-1322.

Singh, Y. and Mandia Harmendra (2011), On composition of generalized fractional integrals involving product of Generalized hyperegeometric functions, International Journal of Physical Sciences, vol. 23(3) A, 727-737.

Singh, Y. and Mandia Harmendra (2011), On the Two Dimensional Weyl Fractional Calculus Associated with the Whittakar Transform, International journal of computer science and Emerging Technology, vol. 2(5), 553-556.

Singh, Y. and Mandia Harmendra (2012), A solution of a convolution integral equation with kernel as a generalized hypergeometric function and *H* -function of two variables, International Journal of Mathematics Research,Vol.4(6),665-670.

Singh, Y. and Mandia Harmendra (2012), An integral equation involving the generalized Struve's function as its kernel with application, International Journal of Mathematics Research, Vol. 4(6), 671-677

Singh, Y. and Mandia Harmendra (2012), On some integrals and Fourier series involving generalized hypergeometric functions, International Journal of Theoretical and Applied Physics (IJTAP), vol.2, No. I, 1-10.

Singh, Y. and Mandia Harmendra (2012), On some multidimensional fractional integral operators associated with Generalized hypergeometric function, International Journal of Engineering Research and Applications (IJERA) vol. 2, Issue 3, 843-849**.**

Singh, Y. and Mandia Harmendra (2012), On some multiplication formulae for generalized hypergeometric functions, International J. of Math. Sci. & Engg. Appls. (IJMSEA), vol. 6 No. II, 39-45.

Singh, Y. and Mandia Harmendra (2012), On the fractional derivative formulae involving the product of a general class of polynomials and the multivariable A –function, IOSR Journal of Mathematics (IOSR-JM) Vol. 3, (1), 46-52

Singh, Y. and Mandia Harmendra (2012), Relationship between Double Laplace Transform and Double Mellin Transform in Terms of Generalized Hypergeometric Function with Applications**,** International Journal of Scientific & Engineering Research vol. 3, Issue 5(1),1-5.

Singh, Y. and Mandia Harmendra (2012),The Relation between Double Laplace Transform and Double Hankel Transform with Applications II, Research & Reviews: Journal of Physics vol. 1, Issue1, 24-30.

Singh, Y. and Mandia Harmendra, On a boundary value problem and its solution involving product of H-function and general class of polynomials, Canadian Journal on Computing in Mathematics, Natural Sciences, Engineering and Medicine, vol.3, No.5,145-152.

Singh, Y. and Mandia, Harmendra (2011), A Unified Study of Astrophysical Thermonuclear Functions for Boltzmann-Gibbs Statistics and TSALLIS Statistics and H- Function , International journal of computer science and Emerging Technology, vol. 2(6), 348-353.

Singh, Y. and Mandia, Harmendra (2011), On some Kober Fractional q-integral operator of the basic analogue of the H- function, International Journal of Theoretical and Applied Physics, vol.1 (1), 53-62.

Singh, Y. and Mandia, Harmendra (2012) ,Convolution integral equation with kernel as a generalized hypergeometric function and H-function of two variables, Canadian Journal on Science and Engineering Mathematics , vol. 3 No. 1,43-47.

Singh, Y. and Mandia, Harmendra (2012), A Unified study of fractional derivative operator in the derivation of bilateral Expansions concerning certain special functions with application, International J. of latest trend in Mathematics, vol.2 , 88-92.

Singh,Y. and Mandia, Harmendra (2013), A study of $\overline{H}$ -function of two variables, International Journal of Innovative research in science, engineering and technology, Vol.2,(9),4914-4921.

Singh and Joshi,(2013). On some double integral transformation of Fox's H-function, Int. J. of Math. Sci.& Engg. Appls., Vol. 7(II), 35-41.

Singh and Malsariya (2014). On Some Double Integrals of H-Function of TwoVariables and Their Applications Journal of Engineering Research and Applications, Vol.4, Issue 8(1),*29-34.*

Singh, R.P. (1989). Generalized Struve's function and its recurrence equations, Vijnana, Parishad Anusandhan Patrika, 28(3), 287-292.

Skibinski, P. (1970). Some expansion theorems for the H -function. Ann. Polon. Math. 23, 125-138.

Sneddon, I. N. (1957). Fourier Transform. McGraw-Hill, New York.

Sneddon, I.N. (1961). Special functions of mathematical physics and chemistry. 2nd ed. Oliver and Boyed, London.

Sneddon, I.N.(1966). Mixed Boundary Value Problems in Potential Theory, North-Holland Publishing Co. , Amsterdam .

Sneddon, I.N. (1972). The uses of Integral Transforms. McGraw-Hill, New York. Srivastava, H.M. and Buschmann R.G.; Composition of fractional integral operators involving Fox's H -function, Acta Mexicana Ci Tecn 7,(1973),21-28.

Soni, R.C. and Singh, D. (2002). Certain fractional derivative formulae involving the product of a general class of polynomials and the multivariable H -function, Proc. Indian Acad. Sci. (Math. Sci.) vol. 112 (4), 551-562.

Springer MD (1979): The Algebra of Random Variables, Wiley, New York**.**

Srivastava, H.M. (1966). Some expansions in products of hypergeometric functions. Proc. Cambridge philos. Soc. 62, 245-247.

Srivastava, H.M. (1967). Generalized Newmann expansions involving hypergeometric functions. Proc. Cambridge philos. Soc. 63, 425-429.

Srivastava, H.M. (1972), A contour integral involving Fox's H -function, Indian J. of Math., 14, 1-6.

Srivastava, H.M. (1972). Some formulas of Hermite and Carlitiz, Rev. Roumaine Math.Pures Appl.,17, 1257-1263.

Srivastava, H.M. and Buschmann R.G.(1973). Composition of fractional integral operators involving Fox's H -function, Acta Mexicana Ci Tecn 7,21-28.

Srivastava, H.M. and Goyal, S.P. (1985); Fractional derivatives of the H -function of several variables, J. Math. Anal. Appl. 112, 641-651.

Srivastava, H.M. and Joshi, C.M. (1967); Certain Double Whittaker transforms of generalized hypergeometric functions, The Yokohama Mathematical Journal, vol. XV, (1) , 17-32.

Srivastava, H.M. and Karlsson, P.W.(1985), Multiple Gaussian Hypergeometric Series, Halssted Press (Ellis Horwood, Chichester),John Wiley and Sons, New York, Chechester and Toronto.

Srivastava, H.M. and Manocha, H.L.(1984). A treatise on generating functions, John Wiley and Sons, New York .

Srivastava, H.M. and Panda, R. (1975). Some analytic or asymptotic confluent expansions for functions of several variables. Math. Comp. 29, 1115-1128.

Srivastava, H.M. and Panda, R. (1976); Some bilateral generating functions for a class of generalized hypergeometric polynomials, J. Reine. Angew Math. 283/28, 265-274.

Srivastava, H.M. and Panda, R. (1976b). Expansion theorems for the H-function of several complex variable. J. Reine Angew. Math. 288, 129-145.

Srivastava, H.M. and Panda, R. (1978). Certain multidimensional integral transformations. I and II. Nederl. Akad. Wetensch. proc. Ser. A81=Indag. Math. 40, 118-131 and 132-144.

Srivastava, H.M. and Panda, R. (1979). Some multiple integral transformations involving the H-function of several variables. Nederl. Akad. Wetensch. Proc, A82=Indag. Math. 41, 353-362.

Srivastava, H.M. and Raina, R.K. (1981). New generating functions for certain polynomial systems associated with the H-function. Hokkaido Math. J. 10, 34-45.

Srivastava, H.M. and Singh, N.P. (1981). The integration of certain products or the multivariable H-function with a general class of polynomials (Abstract No. 81T-33-267). Amer. Math. Soc. Abstracts 2, 404-405.

Srivastava, H.M. and Singh, N.P. (1983). The integration of certain products or the multivariable H function with a general class of polynomials, Rend.circ.Math. Palermo, (Ser.2) 32,157-187.

Srivastava, H.M. and Singh, N.P. (1983); The integration of certain products of the multivariable H-function with a general class of polynomials, Rend. Circ. Mat. Palermo, 32 157-187.

Srivastava, H.M. and Singhal, J.P. (1968),; Double Meijer's transformation of certain hypergeometric functions, Proc. Camb. Phil. Soc. 64.

Srivastava, H.M. and Srivastava, A. (1978). A New Class of orthogonal expansions for the H-function of several complex variables. Comment. Math. Univ. St. Paul 27, 59-69.

Srivastava, H.M., and Joshi, C.M. (1969). Integration of certain products associated with a generalized Meijer function. Proc. Cambridge philos. Soc. 65, 471-477.

Srivastava, H.M., Chandel, R.C. and Vishwakarma, P.K. (1994); Fractional derivatives of certain generalized hypergeometric functions of several variables, J. Math. Anal. Appl. 184, 560-572.

Srivastava, H.M., Goyal, S.P. and Jain, R.N. (1990). Fractional integral operators involving a general class of polynomials. J. Math. Anal. Appl. 148(1). 87-100.

Srivastava, H.M., Gupta, K. C. and Goyal, S. P. (1982): The H-function of one and Two Variables with Applications. South Asian Publishers, New Delhi.

Srivastava, H.M., Koul, C.L. and Raina, R.K. (1981). A class of convolution integral. Univ. of Victoria (Reprint) No. DM- 243-IR.

Srivastava, R. (1981). Definite integrals associated with the H-function of several variables. Comment. Math. Unic. St. Paul 30, 1-5.

Stratton, J.A. (1935). Spheroidal functions, Proc. Nat. Acad. Sci., 21, 51-56.

Suthar, D.L., Saxena, R. K. and Ram, J. (2004).Integral Formulas for the H-functions generalized fractional calculus, South East Asian J. Math. Sc. Vol. 3, pp. 69-74.

Swaroop, R. (1964). A study of Verma transforms collectanes Mathematica de Barcelona, Vol. 26, No. 1, 15-32.

Sxena, R.K. (1974). On a generalized function of n-variables. Kyungpook Math. J. 14, 225-259.

Taha, H.A (1997).Operations Research- An Introduction, 539-550,6/E, Upper Saddle River, N.J.: Prentice Hall.

Takare , N.K. (1972) ; A study of certain sets of orthogonal polynomials and their applications , Ph. D. Thesis, Shivaji Univ. , Kolhapur

Tandon, O.P. (1980). Contiguous relations for the H-function of n- variables. Indian J. pure Appl. Math. 11, 321-325.

Tandon, O.P. (1980a). Finite summation formulas for the H-function of n-variables. Indian J. pure Appl. Math. 11, 23-30.

Tandon, O.P. (1980b). Some finite series concerning the H-function of n-variables. Jnanabha (Professor Arthur Erdelyi Memorial Volume) 9 / 10.

Tsallis C. (1988): Possible generalization of Boltzmann-Gibbs statistics, Journal of Statistical Physics, 52(3), 479-487.

Tsallis, C. (2003). in: Nonextensive Entropy: Interdisciplinary Applications, Eds. M. Gell-Mann, C. Tsallis, Oxford University Press, New York,1

Ueda M, Sergeant AJ, Pato MP and Hussein MS (2001). Effective astrophysical S factor for no resonant reactions, Physical Review C, 61(1), 045801-1:6.

Ueda M, Sergeant AJ, Pato MP and Hussein MS (2004). Resonances and thermonuclear reaction rates for charged particle collisions, Physical Review C., 70(3), 025802-1:6.

Vaishya, G.D., Jain, Renu and Verma, R.C. (1989). Certain properties of the I-function. Proc. Nat. Acad. Sci. India 59(A) II, 329-337.

Vasil'ev,S.S., Kocharov, G.E., Levkovskii , A.A. (1975). Izvestiya Akademii Nauk SSSR, Seriya Fizicheskaya 39, 310.

Vasudevan, T.M. and Shahul Hameed, K.P. (2008). An Integral equation involving the generalized Struve's function as its kernel, The Mathematics Education, vol. 42(2), (2008), 118-122.

Verma, A. (1966). A note on an expansion of hypergeometric functions of two variables. Math. Comp. 29, 413-417.

Verma, R.U. (1971). On the H-function of two variables. II. An. Sti. Univ. 'Al. I. Cuza' Iasi Sect. Ia Mat. (N.S.) 17, 103-109.

Vyas , A.K. (1996). A study of Zonal polynomials and special functions of the real positive definite symmetric matrices, Ph. D. Thesis, J.N.V.Univ., Jodhpur .

Watson, G.N. (1961). A Treaties On The Theory of Bessel Functions, Cambridge Univ. Press., London.

Weyl, H. (1917). Bemerkung zum Begriff des Differential quotienten gebrochener ordenung, Viertel jahrsschrifted. Naturf. Gesellschaft, in Zurich, 62, 296-302.

Whittaker, E.T. and Watson, G.N. (1964). A Course of Modern Analysis, 4th ed. Cambridge.

Wilde, D.J. and Beightler, C.S. (1967). Foundations of Optimization, 31-41, Prentice Hall, Englewood Cliffs, N.J., 1967.

Yadav, R.K. and Purohit, S.D. (2006). On applications of Weyl fractional q-integral operator to generalized basic hypergeometric functions, Kyungpook Math. J. 46: 235-245.

Yadav, R.K. and Purohit, S.D. (2008). On generalized weyl fractional q-integral operator involving generalized basic hypergeometric function, fractional Calculus and Applied Analysis, vol. 11(2),1-14.

Zill, D.G. (1982). A First Course in Differential Equations With Applications, II ed. Prindle, Weber and Bosten.